Damage Mechanics of Composite Materials

Lectures in IIMEC 2012 Summer School on Advanced Composite Materials Technical Educational Institute, Serres, Greece

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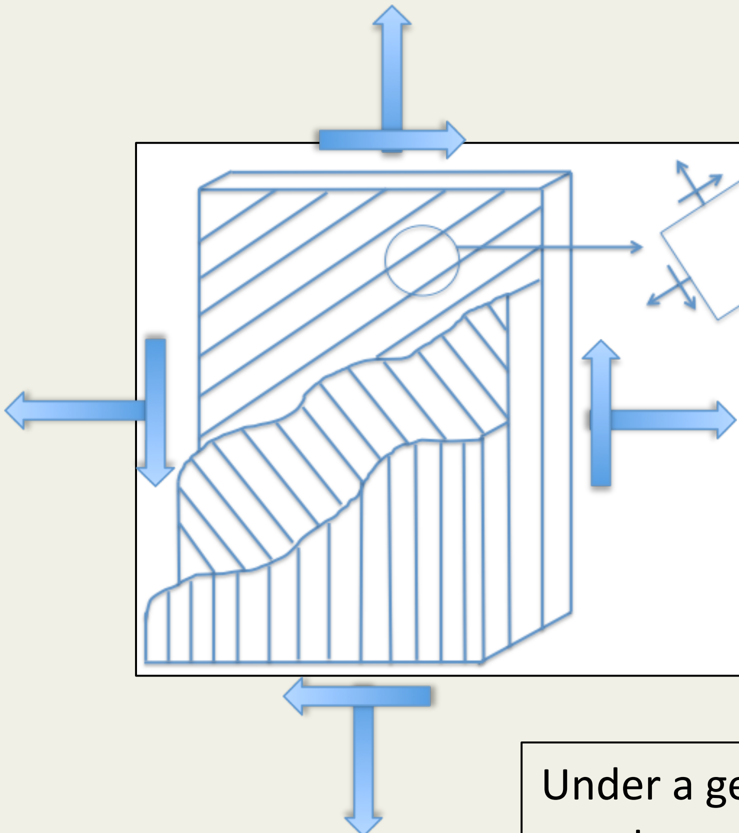
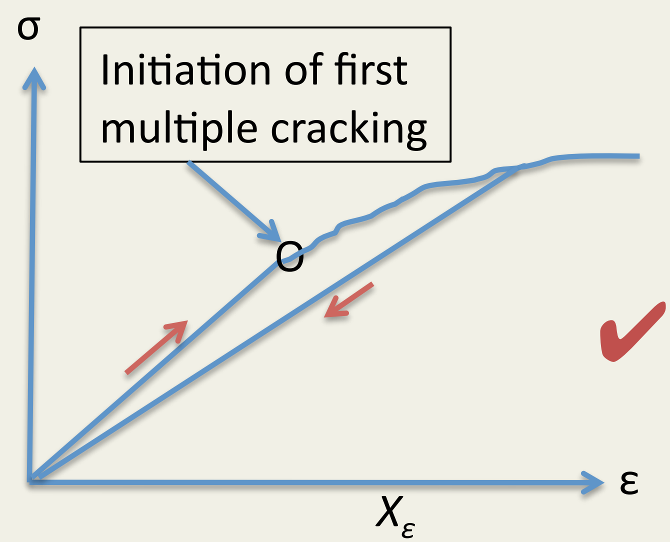
Contents

PART 1: DAMAGE MECHANISMS

Single vs. Multiple Cracking



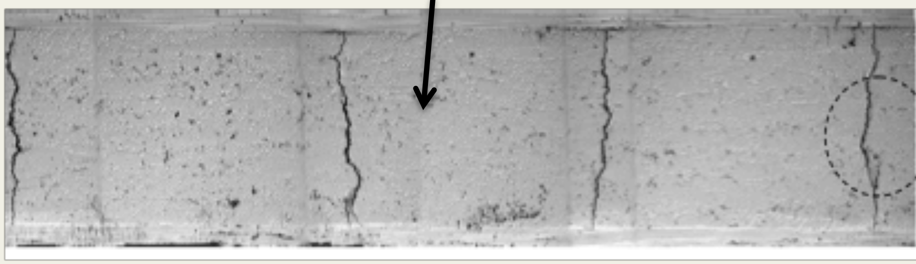


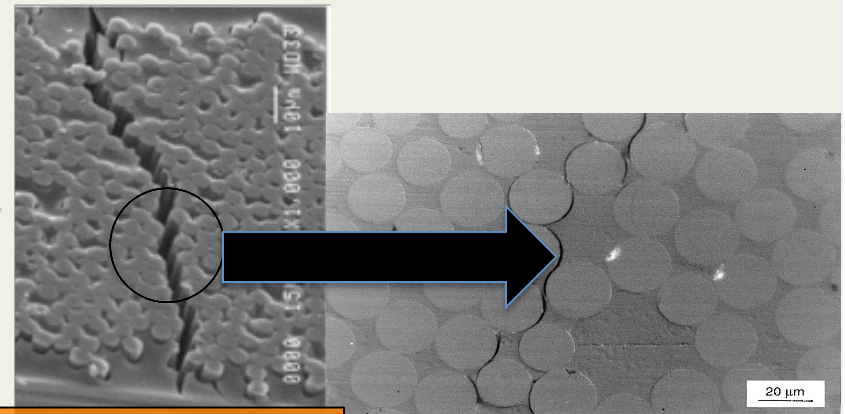
Under a general loading on a laminate, different layers (laminae) are stressed differently.
First crack formation occurs along fibers in a lamina.
On further loading, more cracks form in the same lamina.
This is the MULTIPLE CRACKING process.

Multiple Cracking Process

(a)

Lamina within a laminate



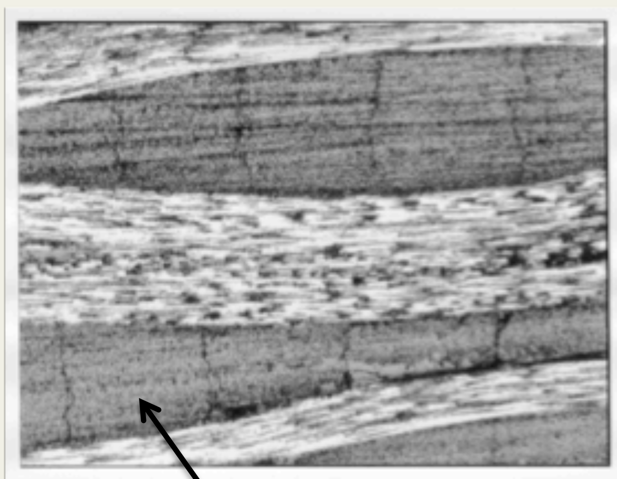


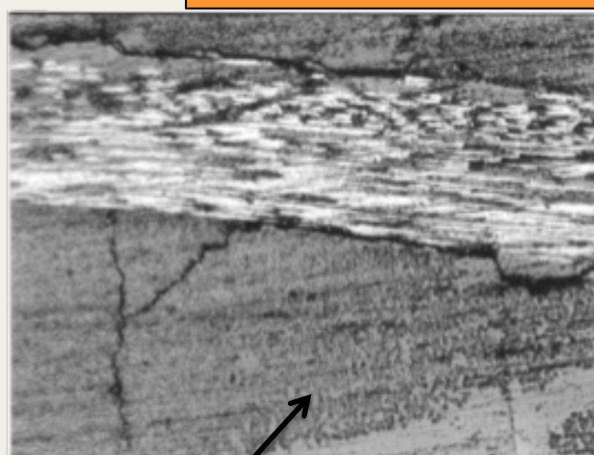
Multiple Cracking

Single crack formation

Fiber/matrix debonding

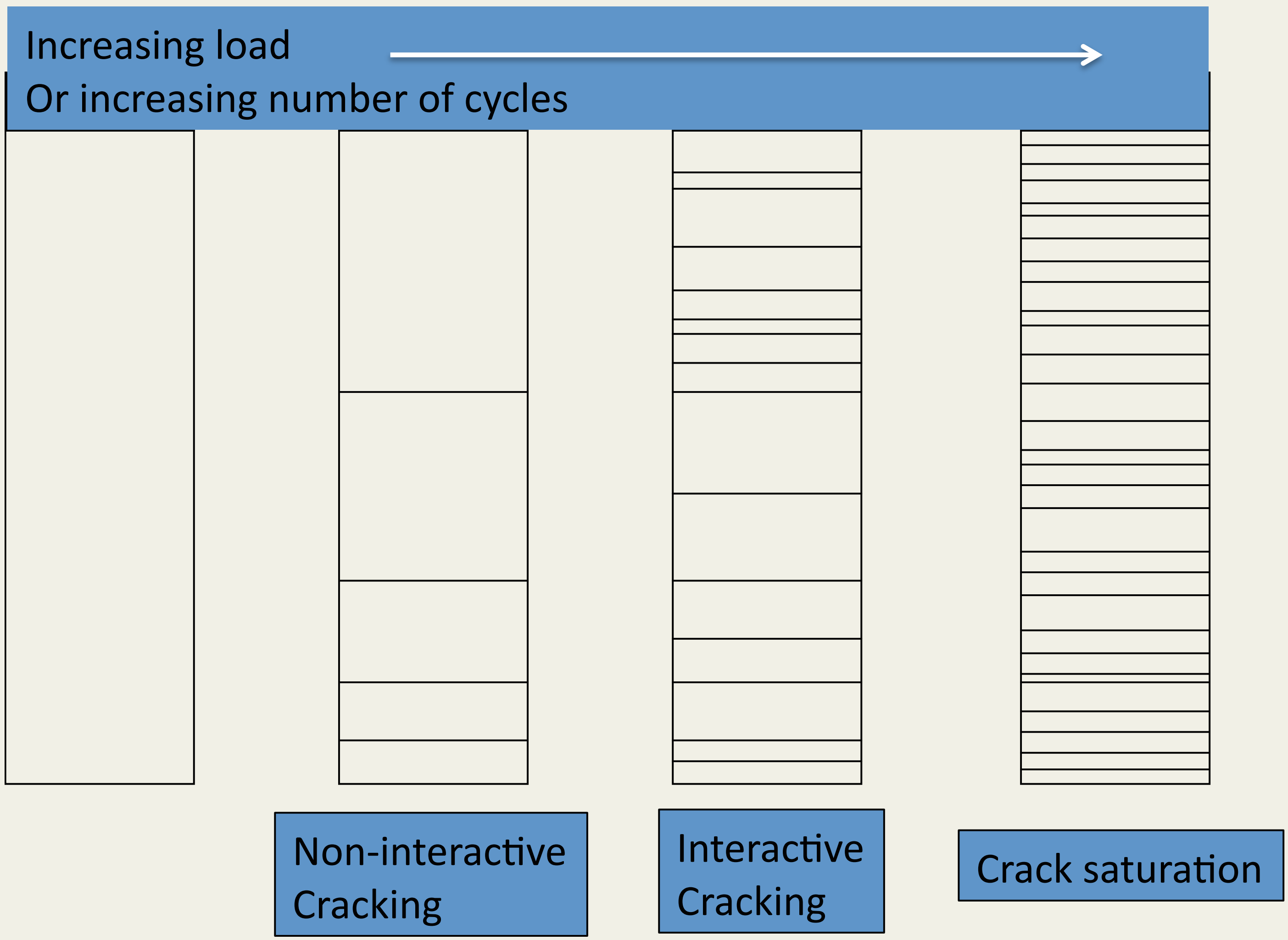
(b)



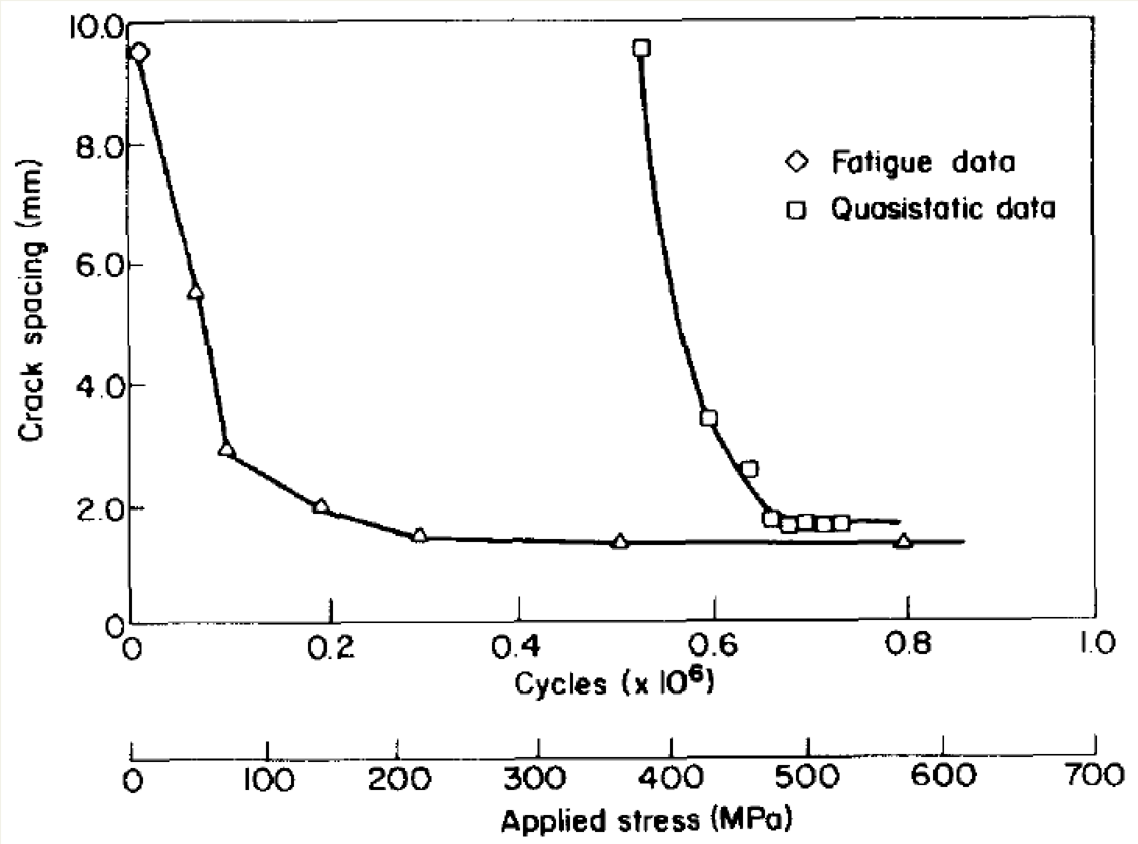


Fiber bundle within a woven fabric composite

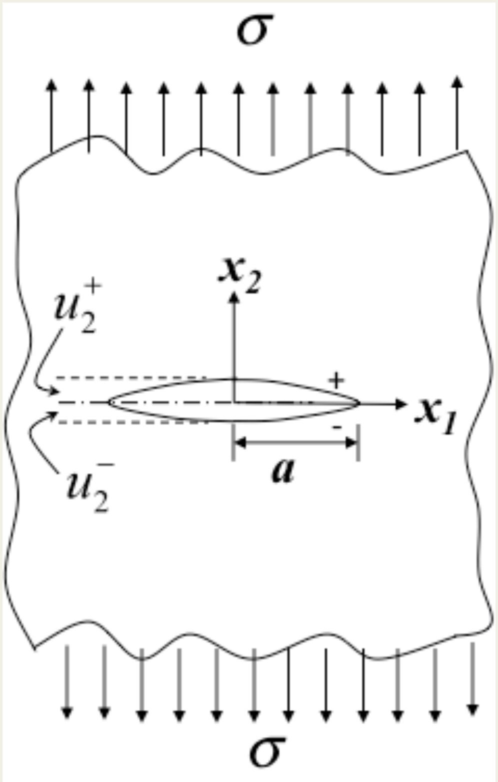
Progression of Multiple Cracking

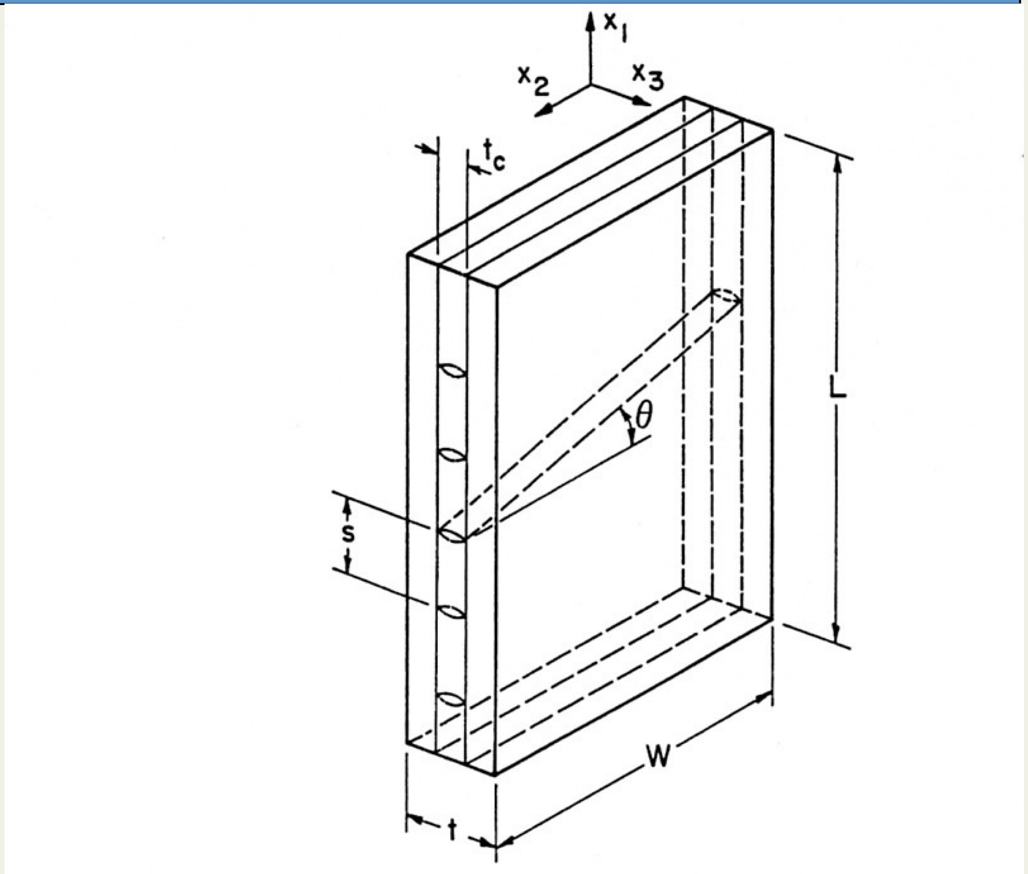


Average Crack Spacing Evolution



Effect on "Constraint" on Multiple Cracking

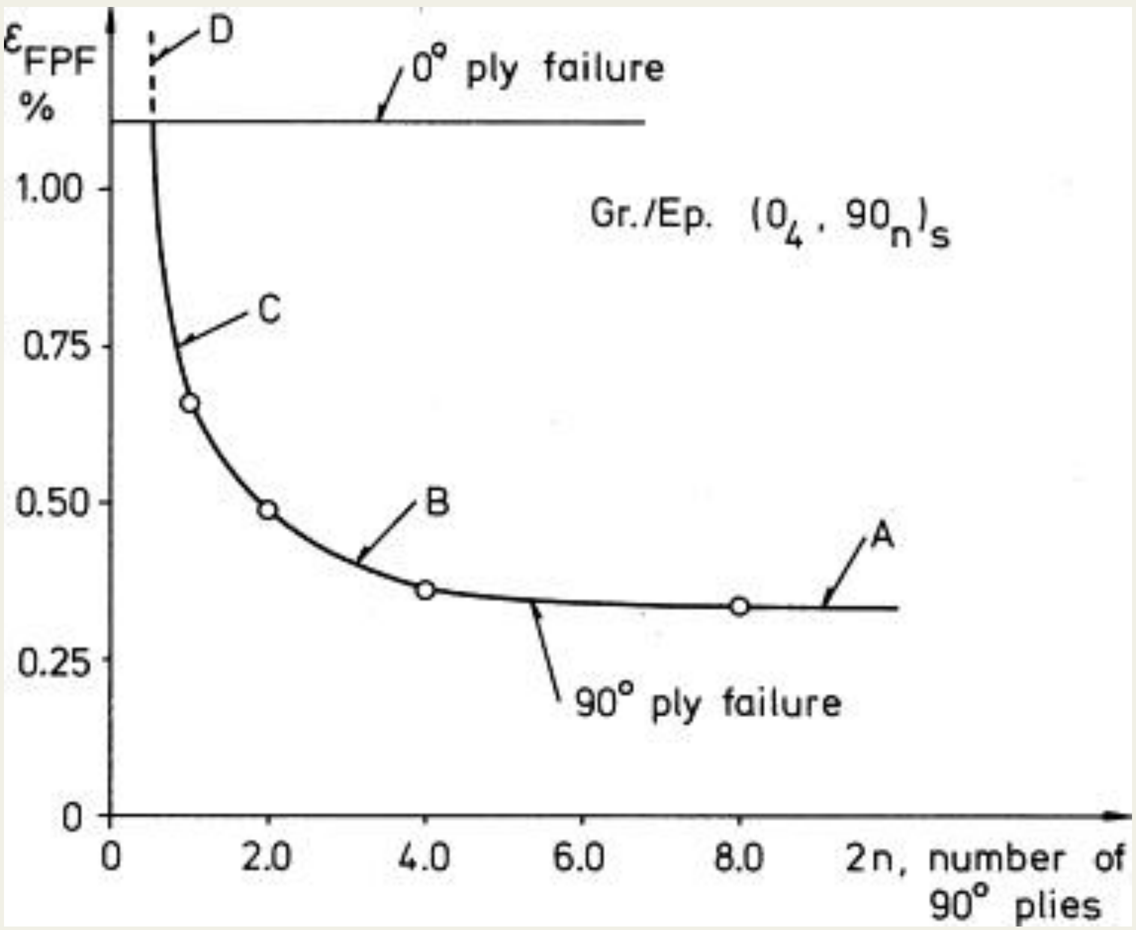


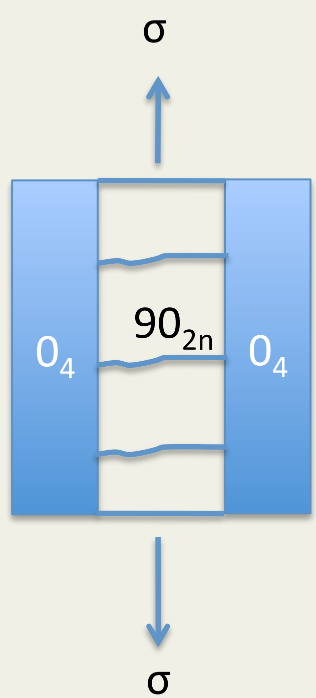


Homogeneous solid:
- Unconstrained crack opening
- Single Fracture (unstable crack growth)

Heterogeneous layered solid (Laminate):
- Constrained crack opening
- Multiple cracking

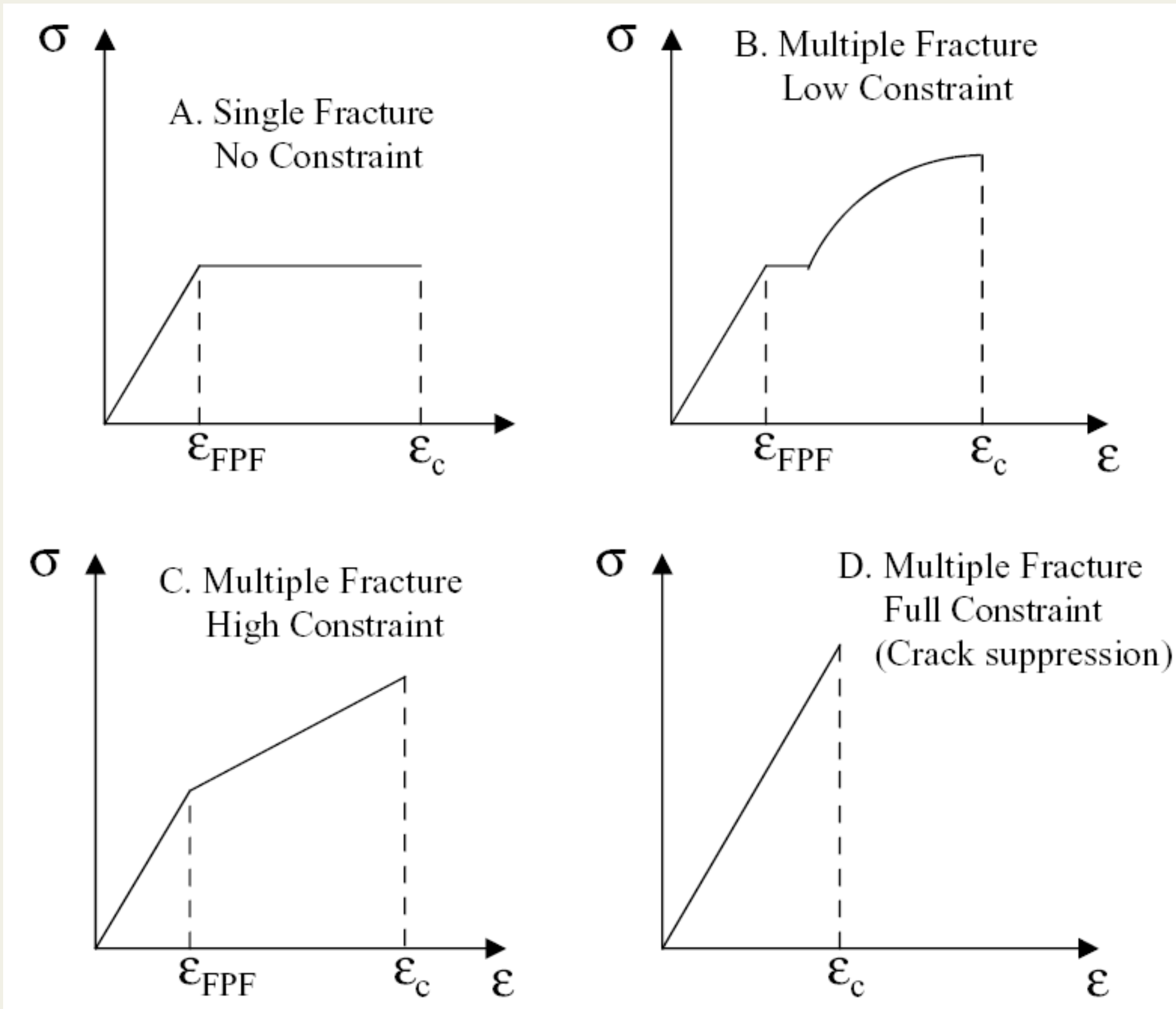
Constraint:
Cracking layer constrained by stiffer layers



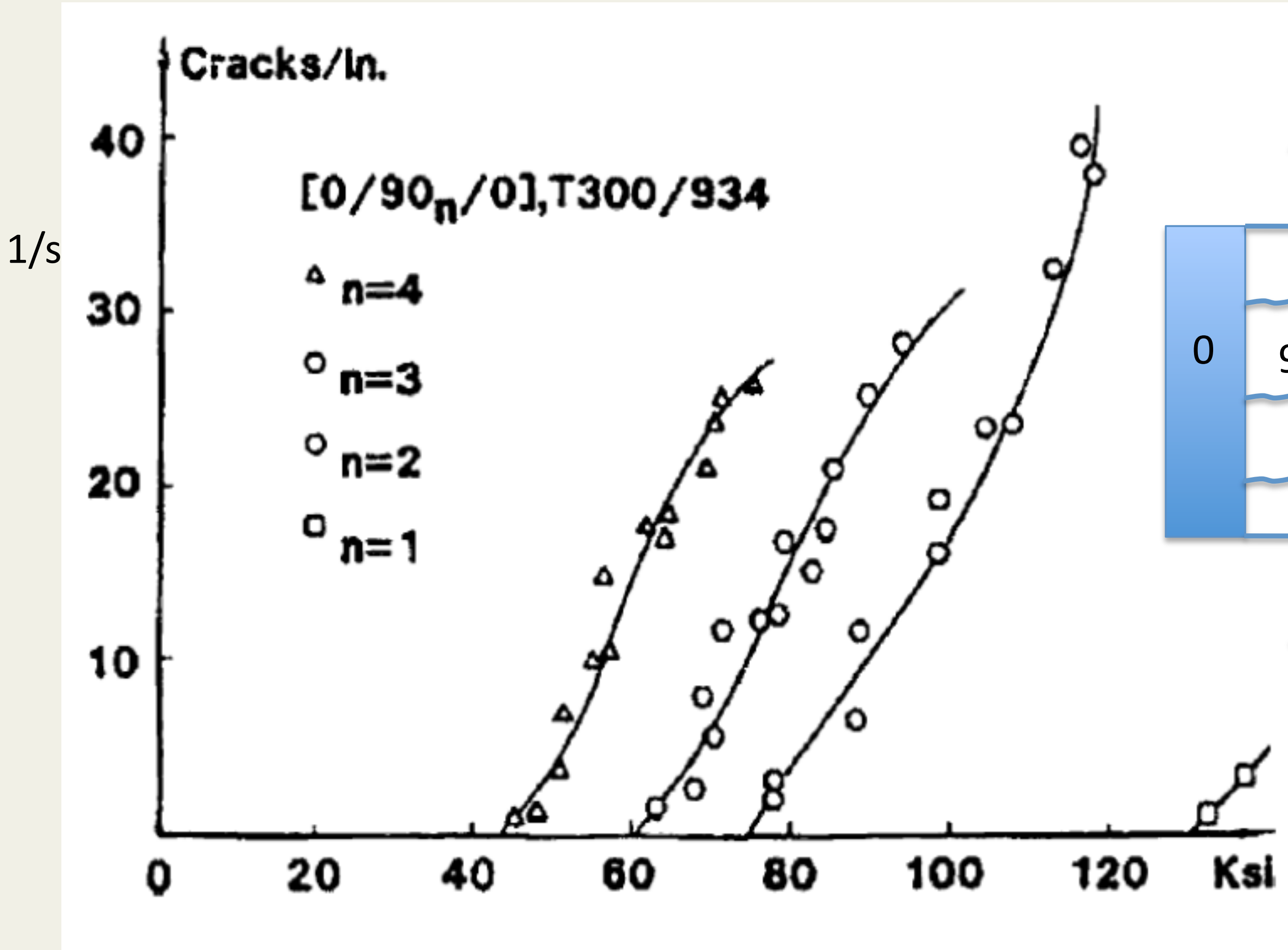


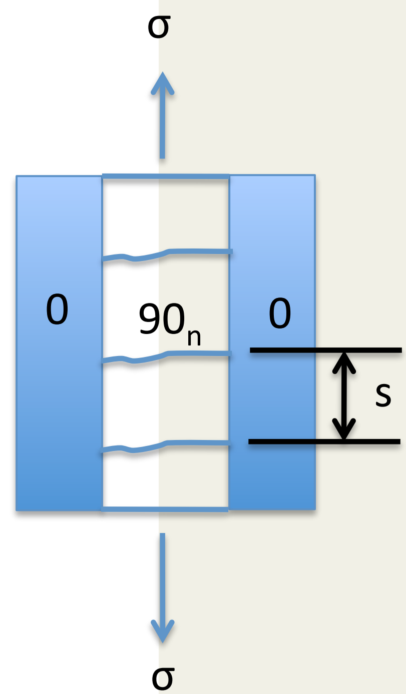
ε_{FPF}: laminate strain at which multiple cracking initiates

Stress-strain response affected by constraint

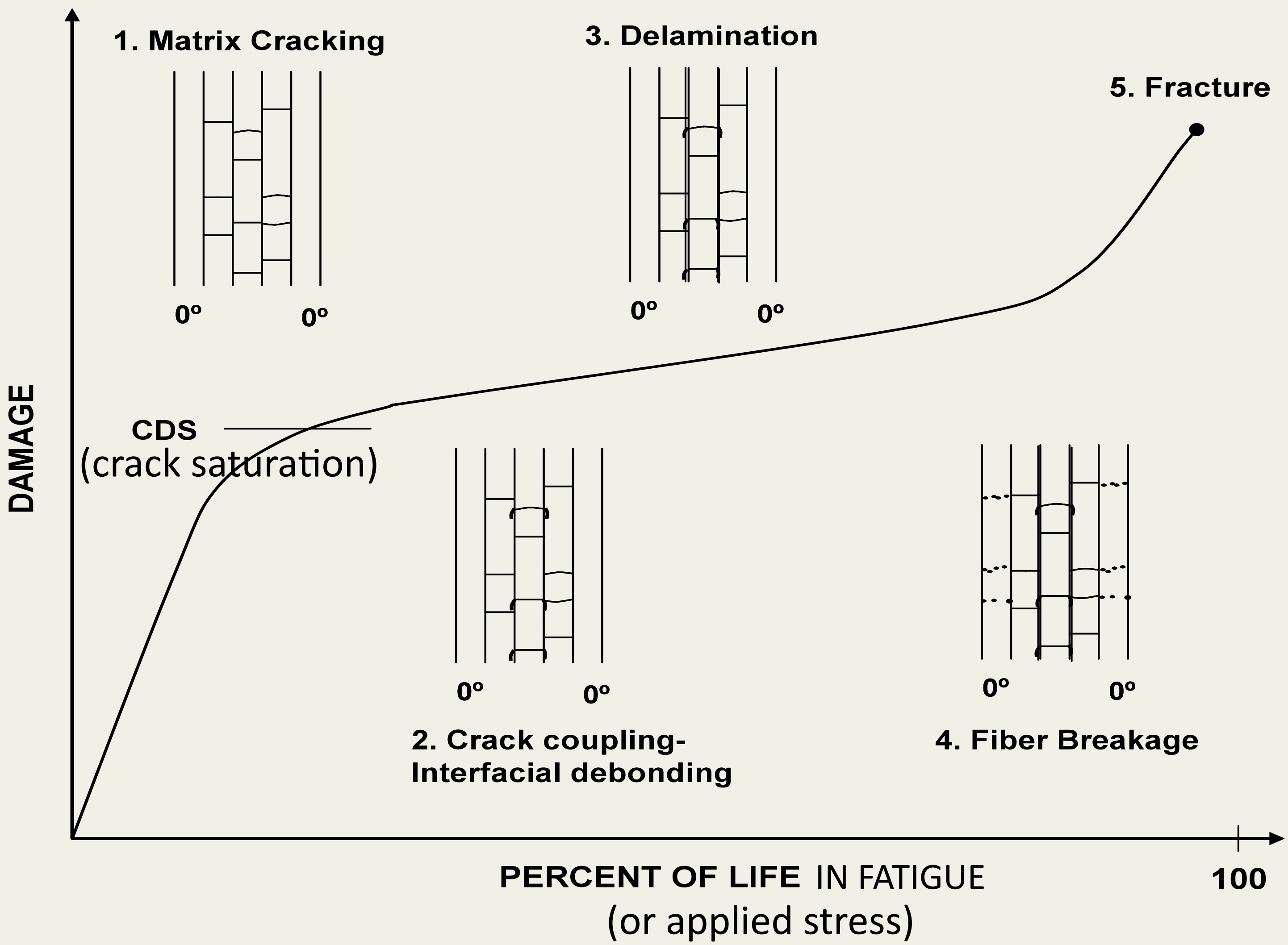


Effect of constraint on evolution of crack density

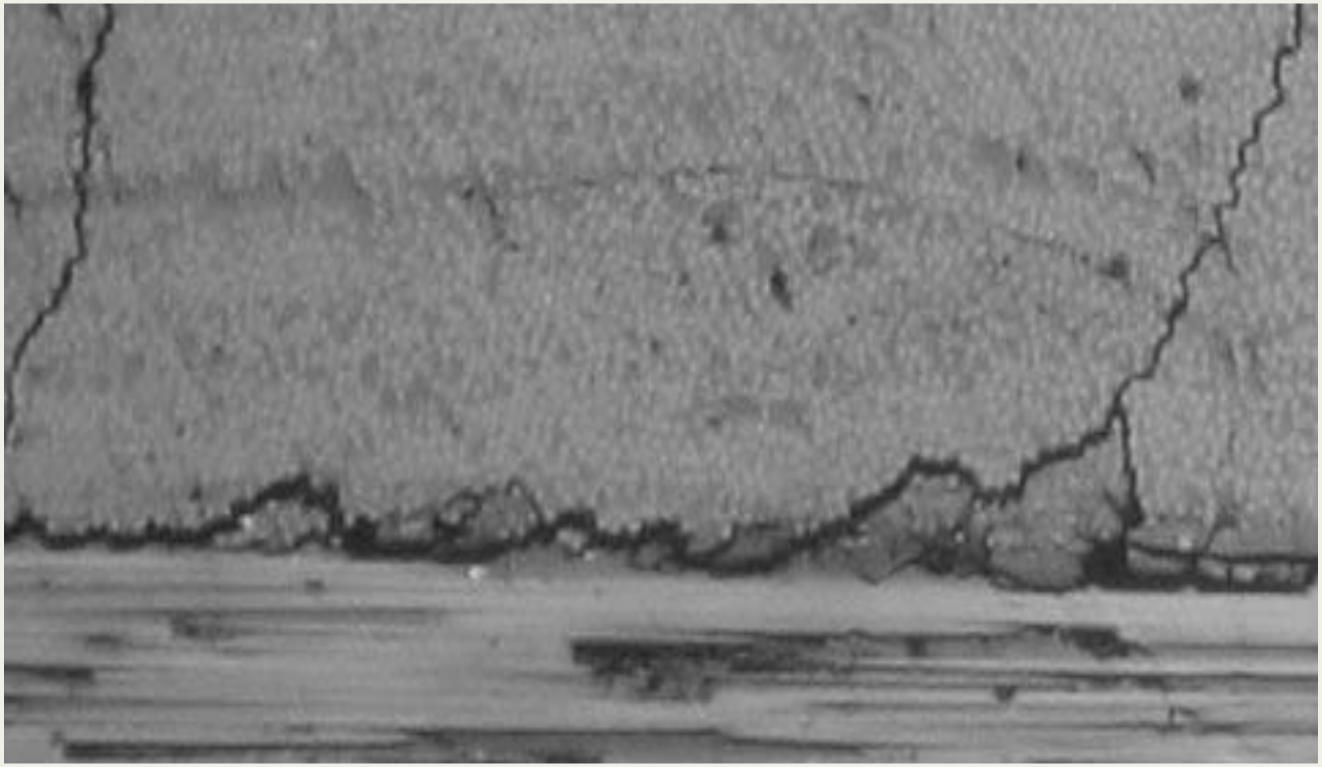




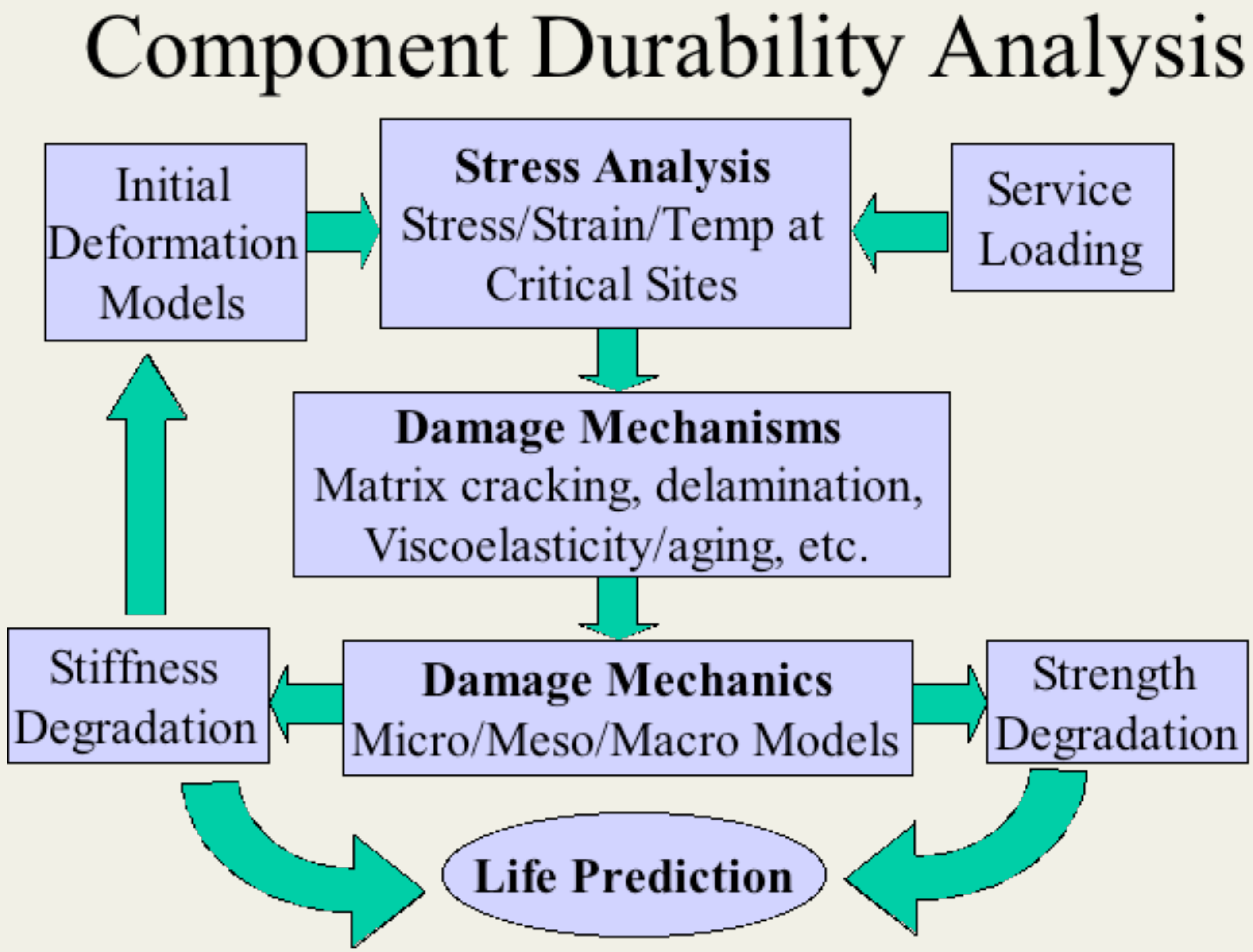
Total Picture of Damage



Crack Coupling at Interfaces



Role of Damage Mechanics



What is Damage Mechanics?

"The subject dealing with mechanics-based analyses of microstructural events in solids responsible for changes in their response to external loading"

Remarks on Damage Mechanics

- Analysis at a single length scale is insufficient
- The hierarchy of microstructural length scales (micro->meso->macro) is not necessarily the same as that for damage
- “Failure” should be defined as “attainment of critical damage state”
- Criticality of damage depends on the type of material performance

Objectives of Damage Mechanics

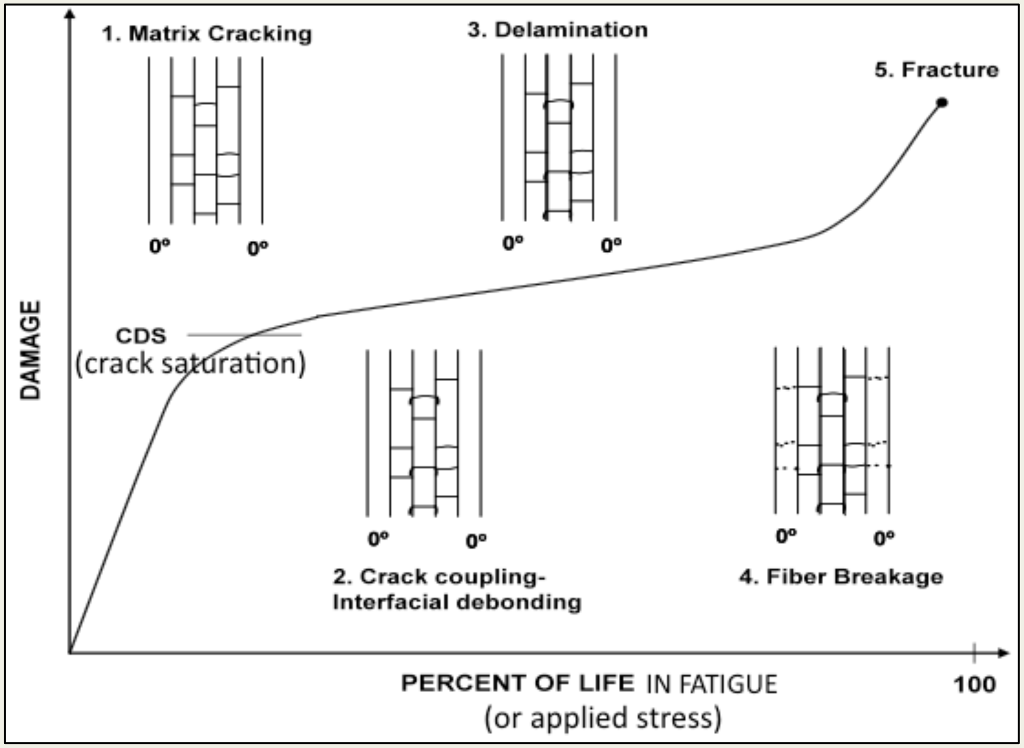
1. Determine the conditions for ***initiation*** of the first damage event
2. Predict the ***evolution*** of progressive damage
3. Characterize and ***quantify*** damage
4. Analyze the effect of damage on material ***response***, e.g. by expressing stiffness properties as a function of damage
5. Define ***criticality*** of damage for assessment of performance (structural integrity and durability)
6. Provide input into overall ***structural analysis*** and design

PART 2:
MICRO-DAMAGE MECHANICS

Desirable Properties of a Damage Modeling Framework

- Should be based on physical mechanisms
- Should have wide applicability (not limited to one or two cases)
- Should be applicable to structural analysis

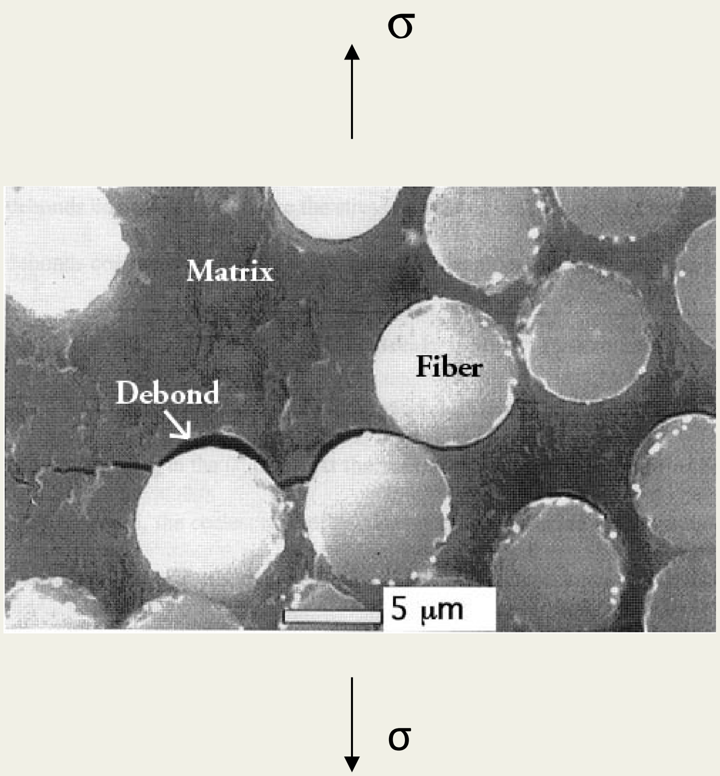
Overview of “Damage” in Composite Laminates



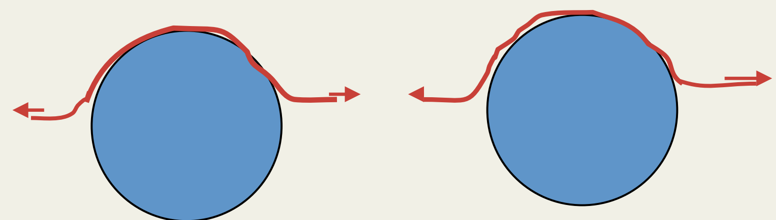
Where to start?
Where to end?

Can one model capture it all?
What are the length scales of damage?

What is the *first* event of damage?

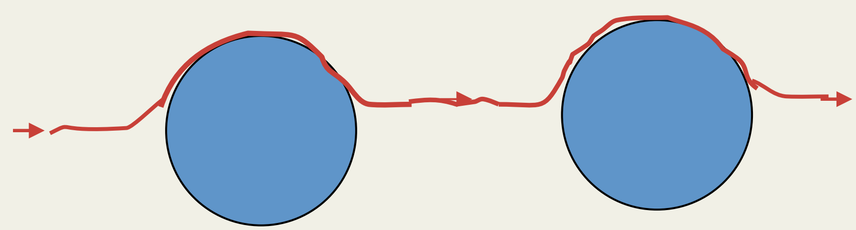


Debonding induces matrix cracking



OR

Matrix cracking causes debonding



Length scales of microstructure:
Fiber diameter, Inter-fiber spacing

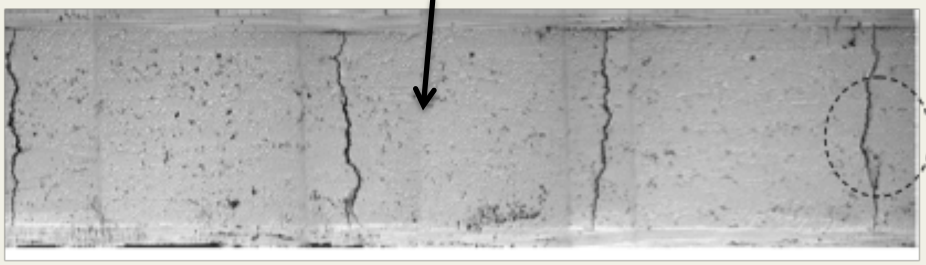
Should we start a model here?

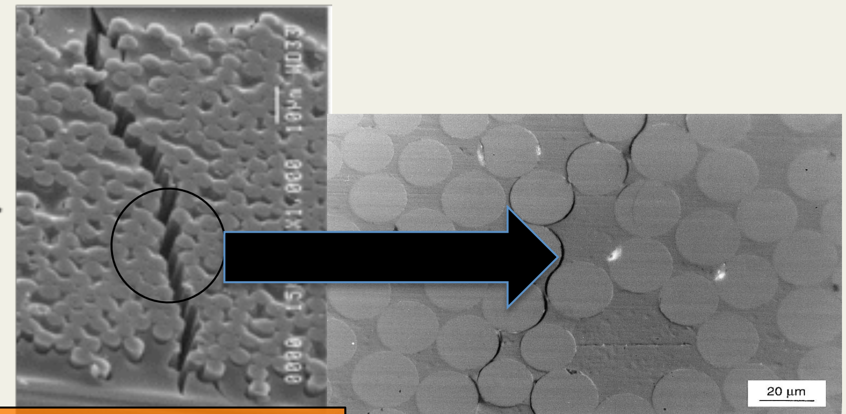
Will it take us to structural failure?

Should we define "damage" as multiple cracking?

(a)

Lamina within a laminate



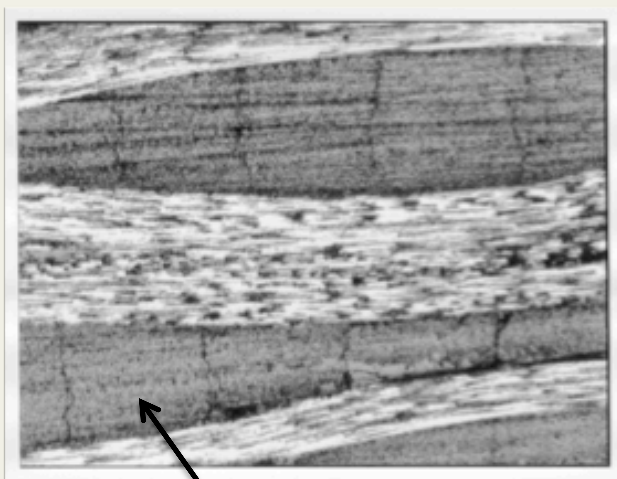


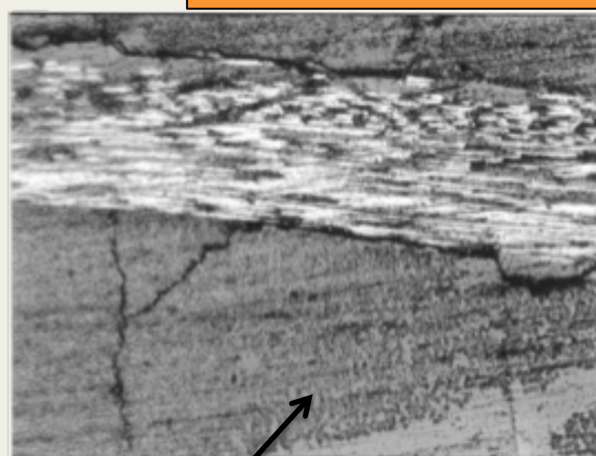
Multiple Cracking

Single crack formation

Fiber/matrix debonding

(b)

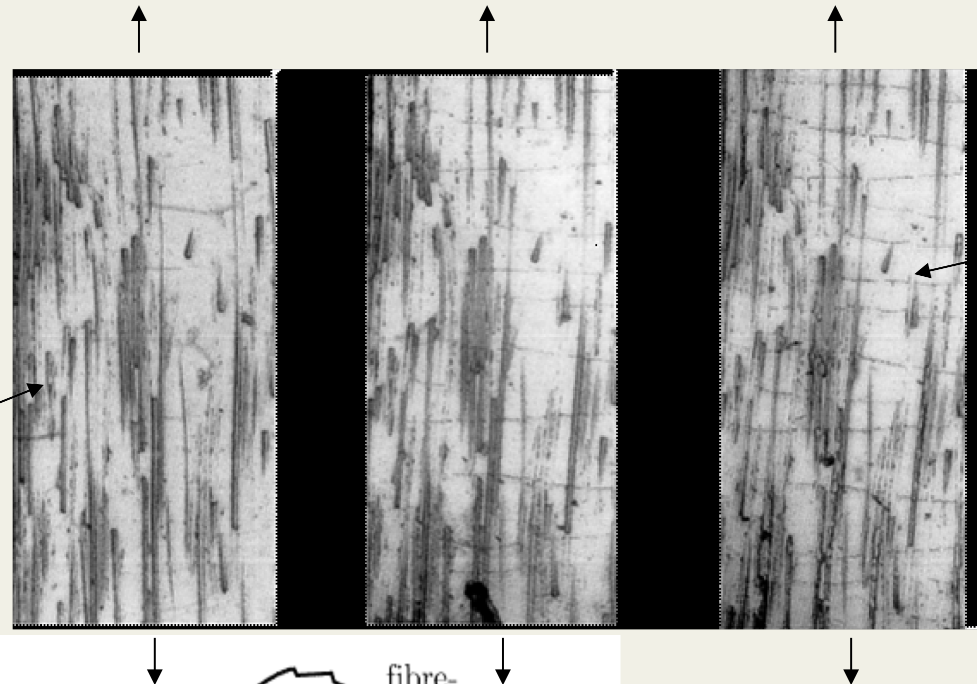




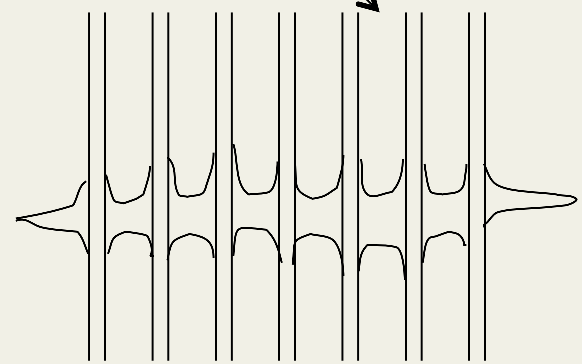
Fiber bundle within a woven fabric composite

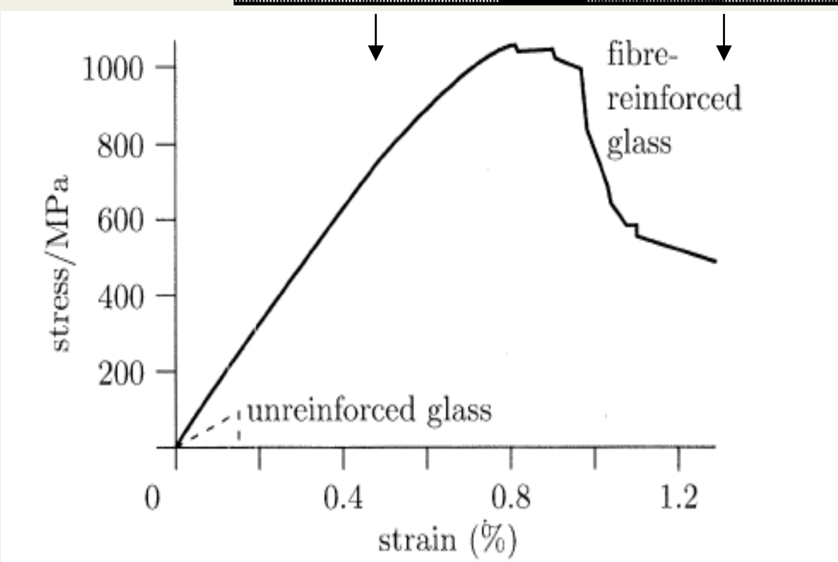
Single vs. Multiple Cracking

Fibers



Fiber-bridged matrix crack





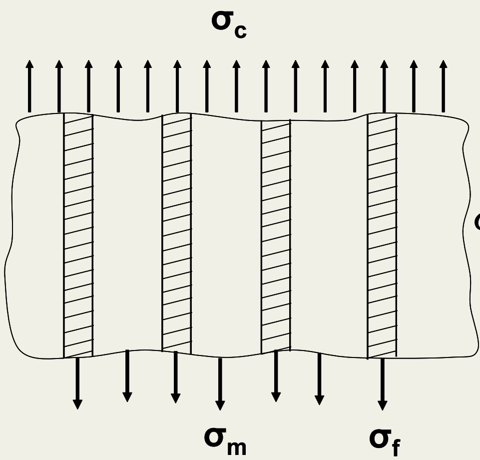
Multiple cracking induces nonlinear stress-strain response and softening

Aveston-Cooper-Kelly (ACK) Theory (1971)

- First satisfactory explanation of multiple cracking
- Approximate one-dimensional stress analysis
- Role of interface in load transfer from cracking (“soft”) constituent to non-cracking (“stiff”) constituent explained by “shear lag”
- Average stiffness reduction (“softening”) of composite predicted

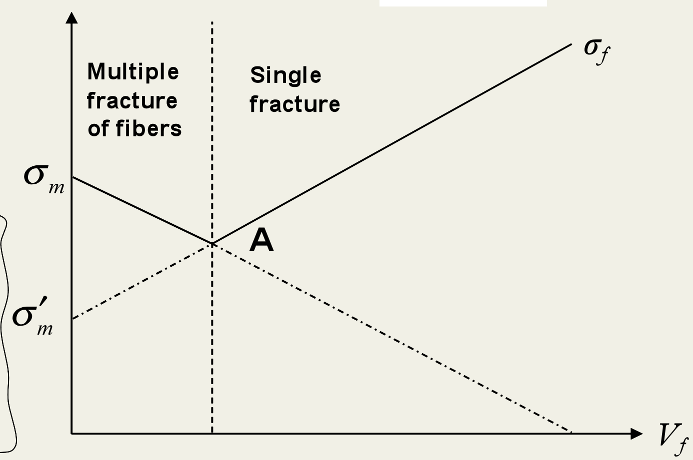
Quick Review of ACK Theory

Conditions for Single vs. Multiple Cracking



$$\sigma_c = \sigma_f V_f + \sigma_m V_m$$

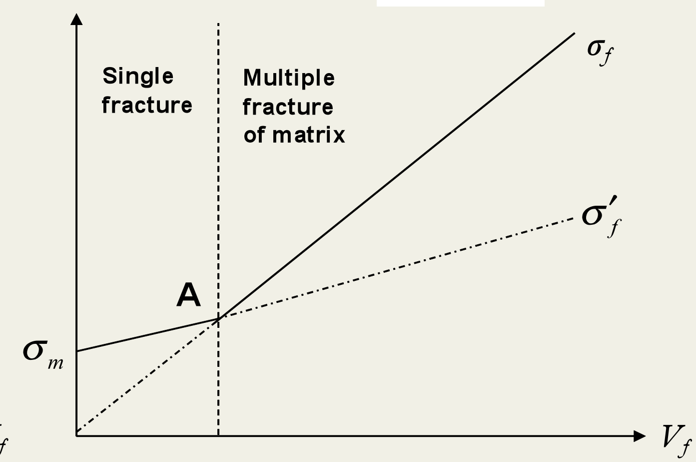
Case 1: $\varepsilon_{fu} < \varepsilon_{mu}$



$$P_{mu} > P_c|_{\varepsilon_{fu}}$$

i.e., if $\sigma_{mu} V_m > \sigma_{fu} V_f + \sigma'_m V_m$

Case 2: $\varepsilon_{mu} < \varepsilon_{fu}$



$$P_{fu} > P_c|_{\varepsilon_{mu}}$$

i.e., if $\sigma_{fu} V_f > \sigma_{mu} V_m + \sigma'_f V_f$

V_f, V_m : Volume fraction of fibers, matrix

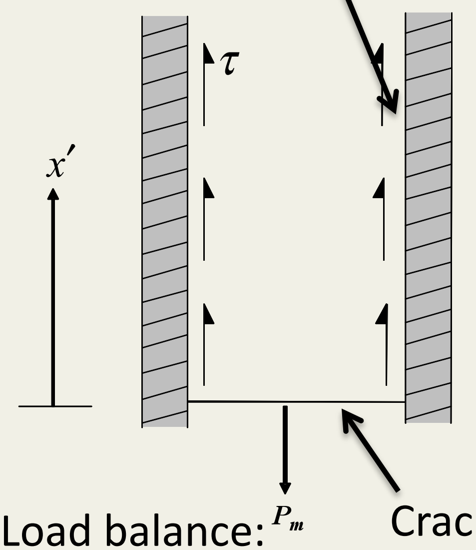
$$\sigma'_m = E_m \varepsilon_{fu} = \frac{\sigma_{mu}}{\varepsilon_{mu}} \varepsilon_{fu}$$

$$\sigma'_f = E_f \varepsilon_{mu} = \frac{\sigma_{fu}}{\varepsilon_{fu}} \varepsilon_{mu}$$

$V_f + V_m = 1$

Quick Review of ACK Theory
Main Results

Complete debonding at interface assumed



Load balance:

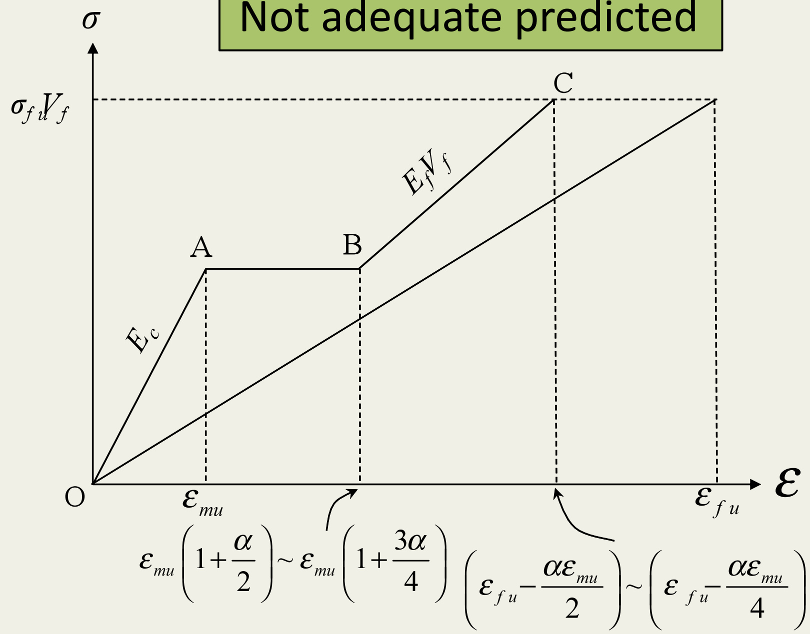
Crack plane

$$P_m = \sigma_{mu} A_m = \tau . 2\pi r . x' . N$$

Distance to next crack:

$$x' = \left(\frac{\sigma_{mu}}{\tau}\right)\frac{V_m}{V_f}\frac{r}{2}.$$

Stress-strain response
Not adequate predicted



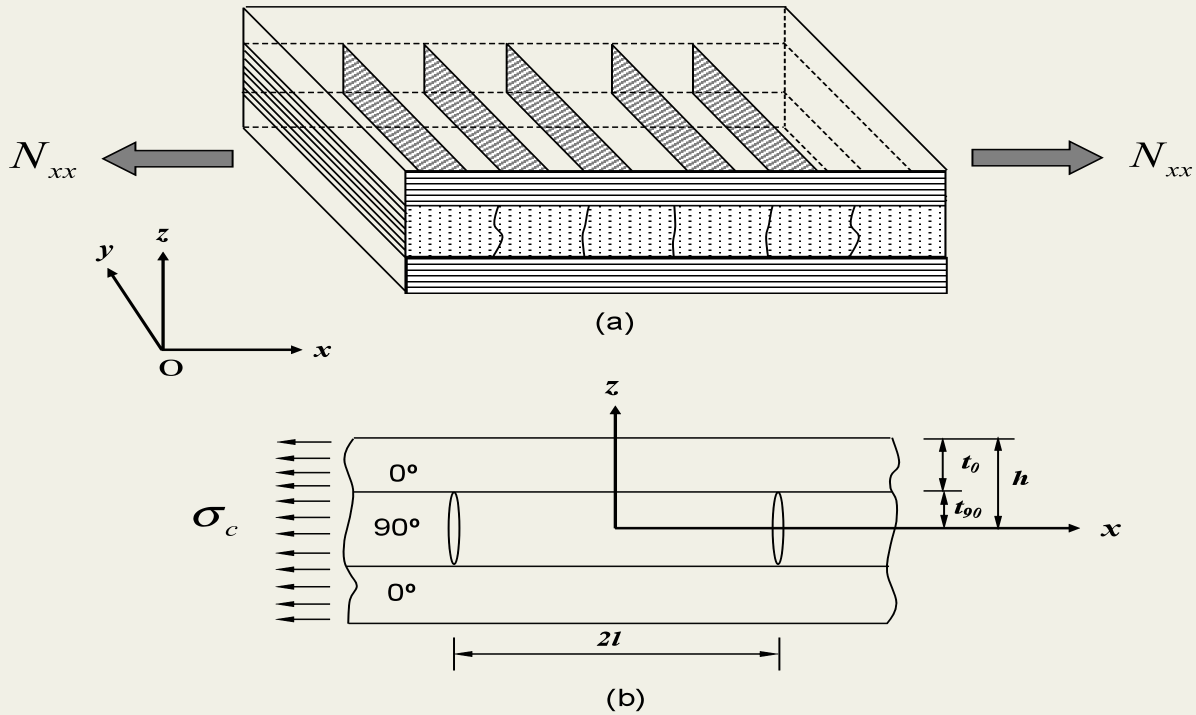
Strain to multiple crack initiation:

$$\varepsilon_{muc} = \left(\frac{12\tau\gamma_m E_f V_f^2}{E_c E_m V_m r}\right)^{1/3}.$$

Υ_m: matrix fracture toughness

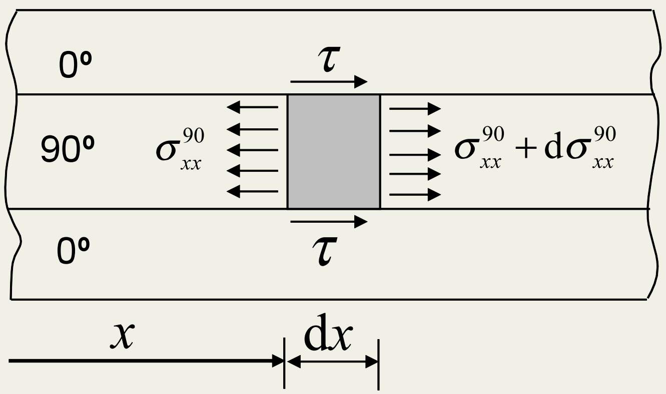
r: fiber radius

Extension to Cross Ply Laminates



- Numerous extensions to cross ply laminates published since ACK (1971)
- Mostly 1-D stress analyses, known as "shear lag" theories
- Various interface assumptions made (shear-lag layer, partial debonding, etc.)
- Only axial stiffness change due to cracking can be predicted

Shear Lag Analysis
Basics



Assumptions:

Shear stress at interface: $\tau = G_{xz}^{90}\left(\frac{u_{90} - u_0}{t_{90}}\right)$

Equilibrium: $\tau = t_{90}\frac{\mathrm{d}\sigma_{xx}^{90}}{\mathrm{d}x}$

Combining, $\frac{\mathrm{d}\sigma_{xx}^{90}}{\mathrm{d}x} = G_{xz}^{90}\left(\frac{u_{90} - u_0}{t_{90}^2}\right).$

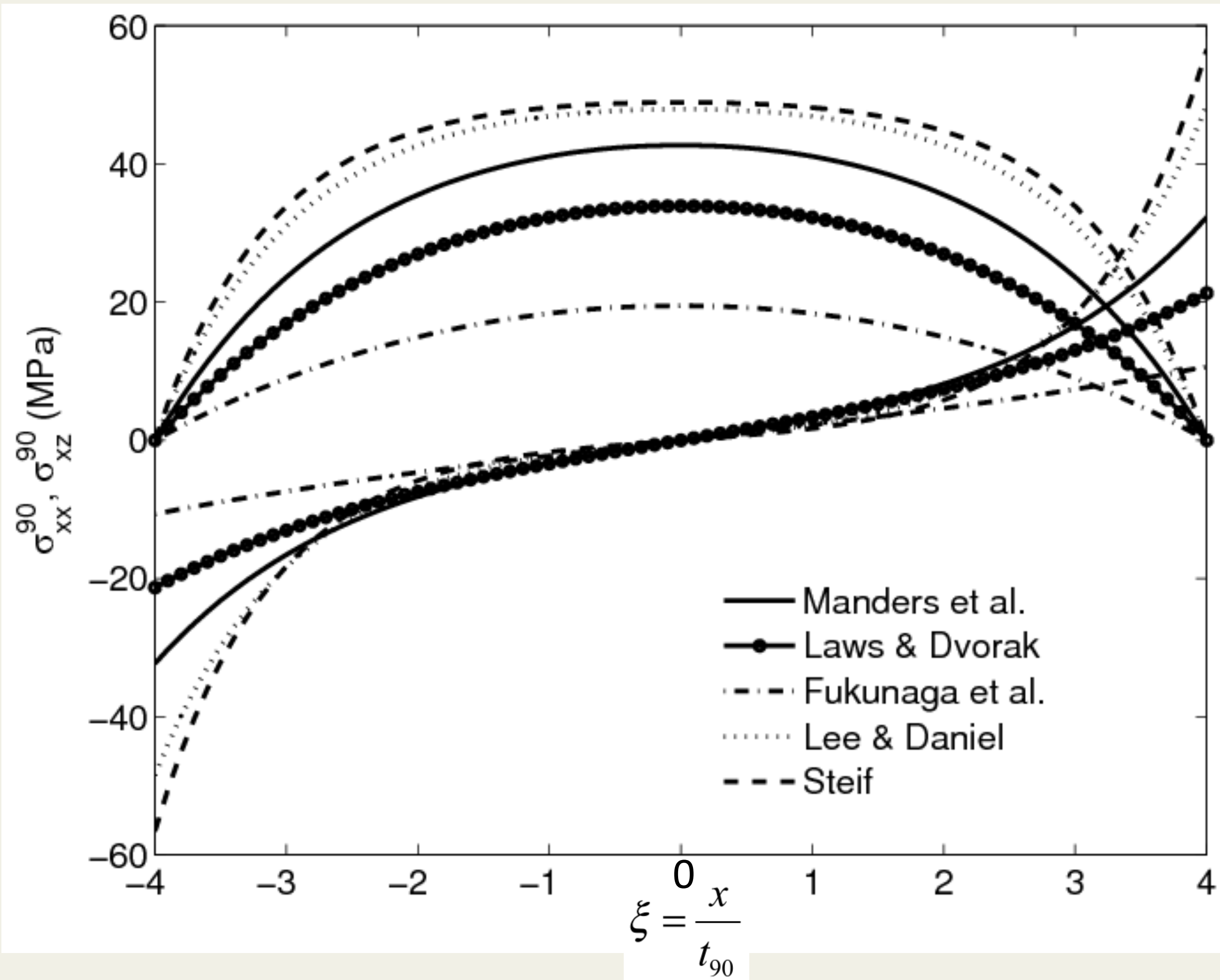
Differentiating, and using axial stress-strain relation, $\frac{\mathrm{d}^2\sigma_{xx}^{90}}{\mathrm{d}x^2} - \frac{\beta^2}{t_{90}^2}\sigma_{xx}^{90} = -\frac{\beta^2}{t_{90}^2}\frac{E_{x0}^{90}}{E_{x0}}\sigma_c$

where $\beta^2 = G_{xz0}^{90}\left[\frac{1}{E_{x0}^{90}} + \frac{1}{\lambda E_{x0}^{0}}\right]$ and $\lambda = \frac{t_0}{t_{90}}.$

β: shear lag parameter, material constant

Solution, axial stress in the cracking ply: $\sigma_{xx}^{90} = \sigma_c\frac{E_{x0}^{90}}{E_{x0}}\left(1 - \frac{\cosh\beta\frac{x}{t_{90}}}{\cosh\beta\frac{l}{t_{90}}}\right).$

Axial Normal and Shear Stress Predictions

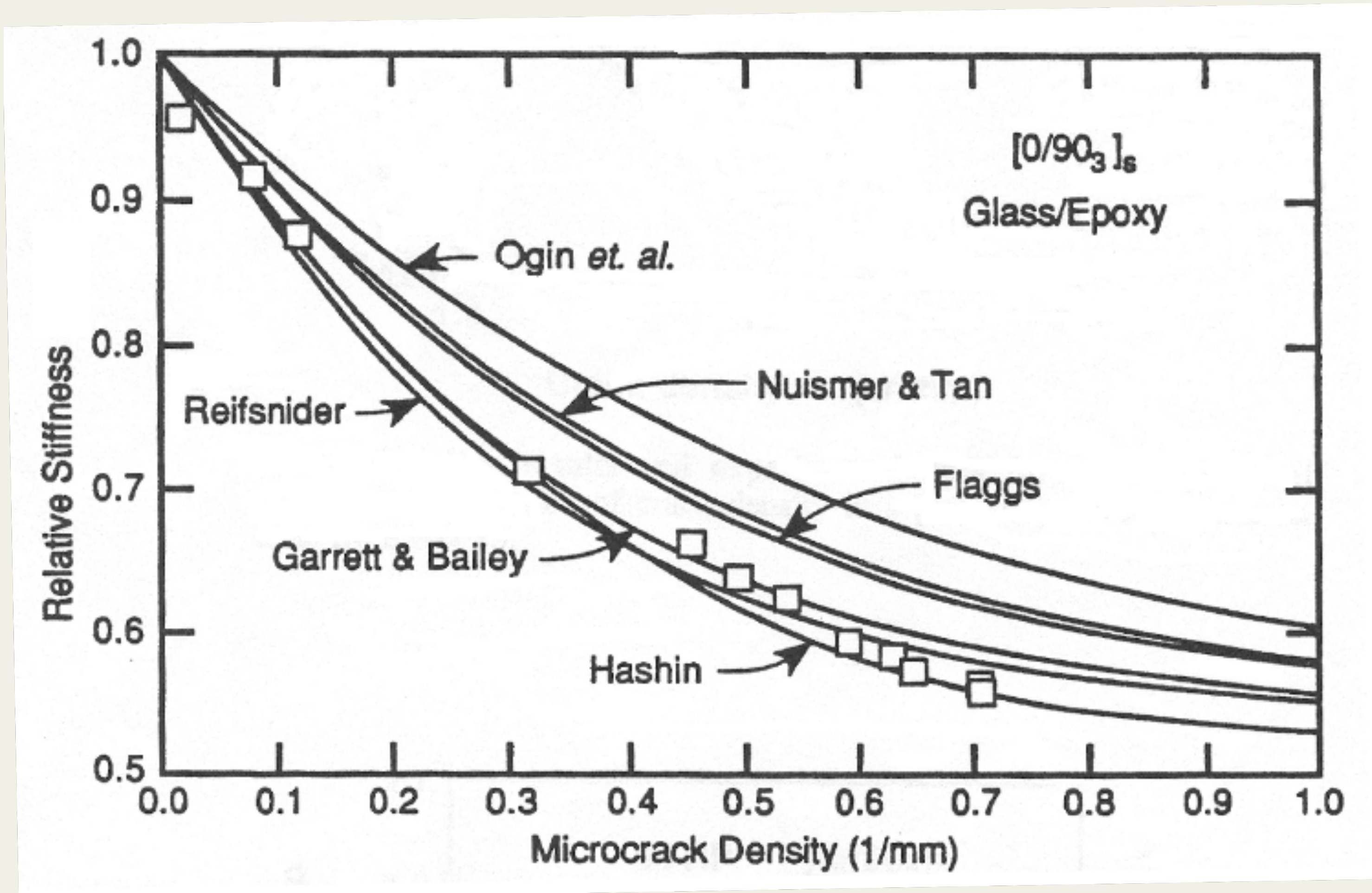


No crack interactions

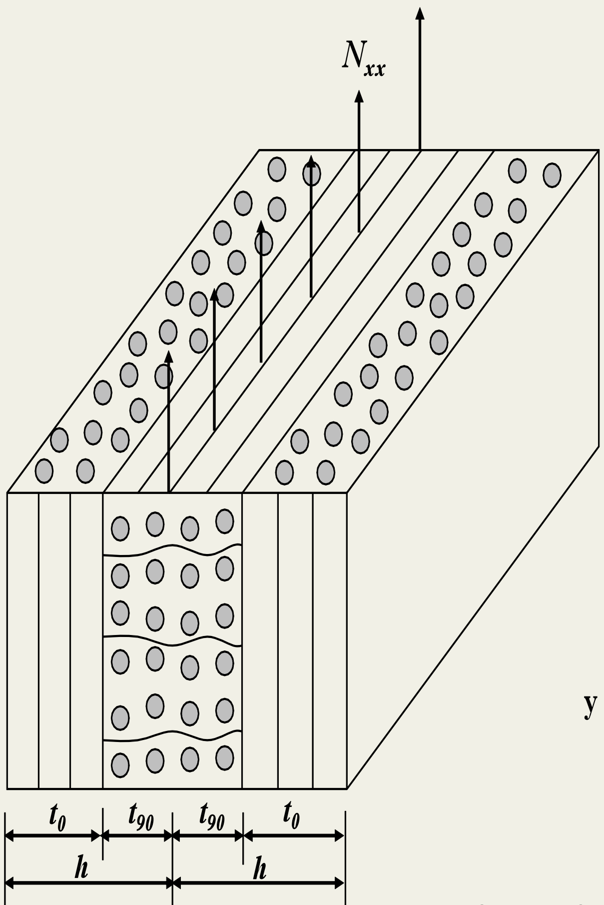
Incorrect shear stress at crack planes

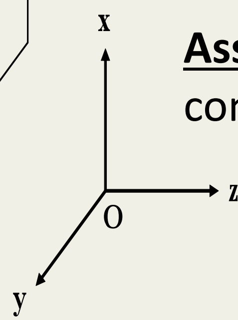
Predictions depend on assumed shear lag parameter

Prediction of Average Axial Stiffness (Other properties cannot be predicted)



Next Advance in Stress Analysis
2-D Variational Model, Hashin (1985)





Stresses before cracking:

$$\sigma^{0}_{xx0}, \sigma^{90}_{xx0} \neq 0, \qquad \sigma^{0}_{yy0}, \sigma^{90}_{yy0} \neq 0$$
$$\sigma^{0}_{zz0} = \sigma^{90}_{zz0} = 0, \qquad \sigma^{0}_{yz0} = \sigma^{90}_{yz0} = 0$$
$$\sigma^{0}_{xz0} = \sigma^{90}_{xz0} = 0, \qquad \sigma^{0}_{xy0} = \sigma^{90}_{xy0} = 0$$

$$\sigma^{0}_{xx0} = \frac{E^{0}_{x0}}{E_{x0}}\sigma_c, \sigma^{90}_{xx} = \frac{E^{90}_{x0}}{E_{x0}}\sigma_c$$
$$\sigma_c = \frac{N_{xx}}{2h}$$

Assume: Crack-induced perturbations in axial stresses constant in thickness (z) direction

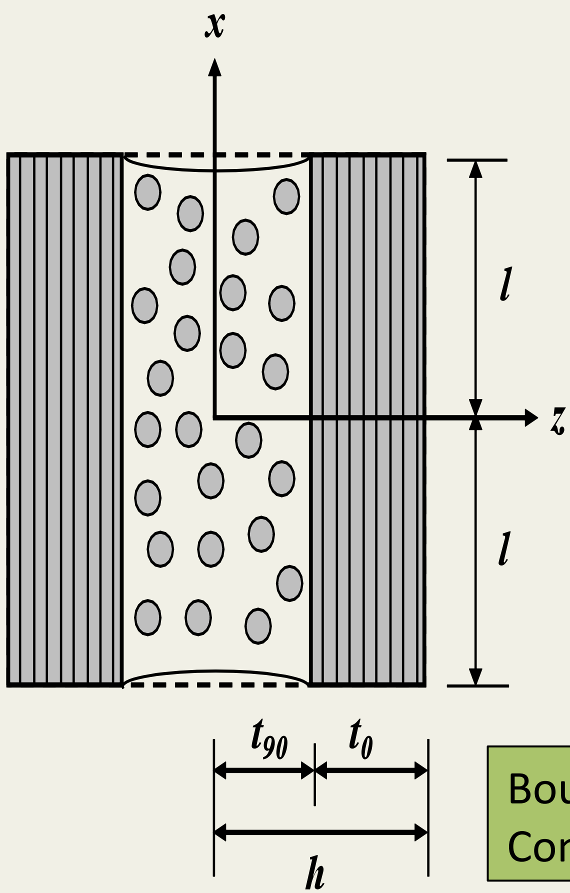
$$\Delta\sigma^{90}_{xx} = \Delta\sigma^{90}_{xx}(x) \qquad \Delta\sigma^{0}_{xx} = \Delta\sigma^{0}_{xx}(x)$$

Or

$$\Delta\sigma^{90}_{xx} = -\sigma^{90}_{xx0}\phi_{90}(x)$$
$$\Delta\sigma^{0}_{xx} = -\sigma^{0}_{xx0}\phi_{0}(x)$$

Where by axial force balance $\phi_0(x) = -\frac{\sigma^{90}_{xx0}}{\sigma^{0}_{xx0}}\frac{1}{\lambda}\phi_{90}(x)$ $\quad \lambda = \frac{t_0}{t_{90}}$

Boundary Value Problem Formulation



Equilibrium equations:

$$\frac{\partial \sigma_{xx}^{m}}{\partial x}+\frac{\partial \sigma_{xy}^{m}}{\partial y}+\frac{\partial \sigma_{xz}^{m}}{\partial z}=0$$

$$\frac{\partial \sigma_{yx}^{m}}{\partial x}+\frac{\partial \sigma_{yy}^{m}}{\partial y}+\frac{\partial \sigma_{yz}^{m}}{\partial z}=0$$

$$\frac{\partial \sigma_{zx}^{m}}{\partial x}+\frac{\partial \sigma_{zy}^{m}}{\partial y}+\frac{\partial \sigma_{zz}^{m}}{\partial z}=0$$

Ply stresses

$$\sigma_{ij}^{m}=\sigma_{ij0}^{m}+\Delta\sigma_{ij}^{m}$$

$$\frac{\partial \Delta\sigma_{xx}^{m}}{\partial x}+\frac{\partial \Delta\sigma_{xz}^{m}}{\partial z}=0$$

$$\frac{\partial \Delta\sigma_{zx}^{m}}{\partial x}+\frac{\partial \Delta\sigma_{zz}^{m}}{\partial z}=0$$

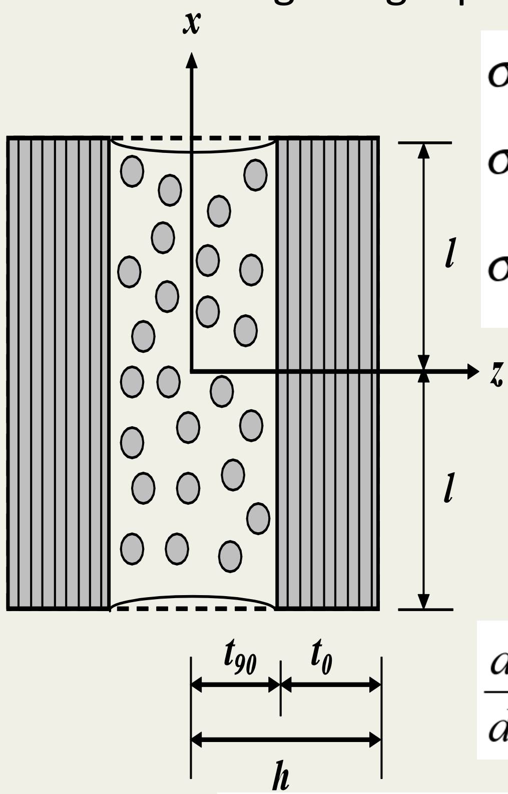
m = 0 (0-deg ply); m = 90 (90-deg ply)

Boundary Conditions

Symmetry:	$\sigma_{xz}^{90}(x,0)=0$
Interface	$\sigma_{xz}^{90}(x,t_{90})=\sigma_{xz}^{0}(x,t_{90})$
	$\sigma_{zz}^{90}(x,t_{90})=\sigma_{zz}^{0}(x,t_{90})$
Free boundary:	$\sigma_{xz}^{0}(x,h)=0$
	$\sigma_{zz}^{0}(x,h)=0$
Traction free	$\sigma_{xz}^{90}(\pm l,z)=0, \quad -t_{90}\le z\le t_{90}$
	$\sigma_{xx}^{90}(\pm l,z)=0, \quad -t_{90}\le z\le t_{90}$

Boundary Value Problem Solution

Integrating equilibrium equations, and applying boundary conditions, give:



$$\sigma_{xx}^{90} = \sigma_{xx0}^{90}\left[1-\phi(x)\right]$$

$$\sigma_{xz}^{90} = \sigma_{xx0}^{90}\phi'(x)z$$

$$\sigma_{zz}^{90} = \sigma_{xx0}^{90}\phi''(x)\frac{1}{2}\left[(1+\lambda)t_{90}^2 - z^2\right]$$

$$\sigma_{xx}^{0} = \sigma_{xx0}^{0} + \sigma_{xx0}^{90}\frac{1}{\lambda}\phi(x)$$

$$\sigma_{xz}^{0} = \sigma_{xx0}^{90}\phi'(x)\frac{1}{\lambda}\left[(1+\lambda)t_{90} - z\right]$$

$$\sigma_{zz}^{0} = \sigma_{xx0}^{90}\phi''(x)\frac{1}{2\lambda}\left[(1+\lambda)t_{90} - z\right]^2$$

The unknown perturbation function ϕ(x) found by applying the principle of minimum complementary energy.
Euler-Lagrange equation for ϕ :

$$\frac{d^4\phi}{d\xi^4} + p\frac{d^2\phi}{d\xi^2} + q\phi = 0 \;\Rightarrow\; \phi = A_1 \cosh\alpha_1\xi \cos\alpha_2\xi + A_2 \sinh\alpha_1\xi \sin\alpha_2\xi$$

$$\xi = \frac{x}{t_{90}}$$

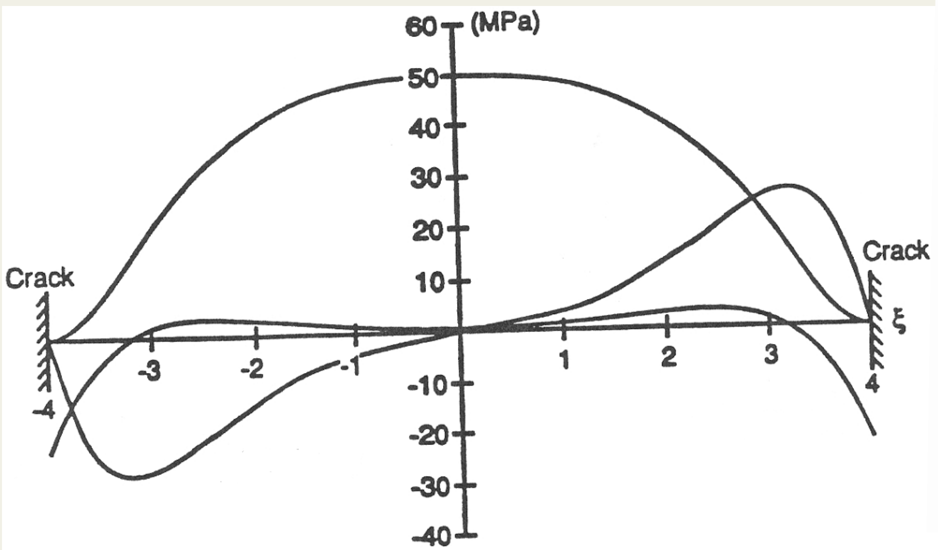
$$A_1 = \frac{2\left(\alpha_1 \cosh\alpha_1\rho \sin\alpha_2\rho + \alpha_2 \sinh\alpha_1\rho \cos\alpha_2\rho\right)}{\alpha_1 \sin 2\alpha_1\rho + \alpha_2 \sinh 2\alpha_2\rho}$$

$$A_2 = \frac{2\left(\alpha_2 \cosh\alpha_1\rho \sin\alpha_2\rho - \alpha_1 \sinh\alpha_1\rho \cos\alpha_2\rho\right)}{\alpha_1 \sin 2\alpha_1\rho + \alpha_2 \sinh 2\alpha_2\rho}$$

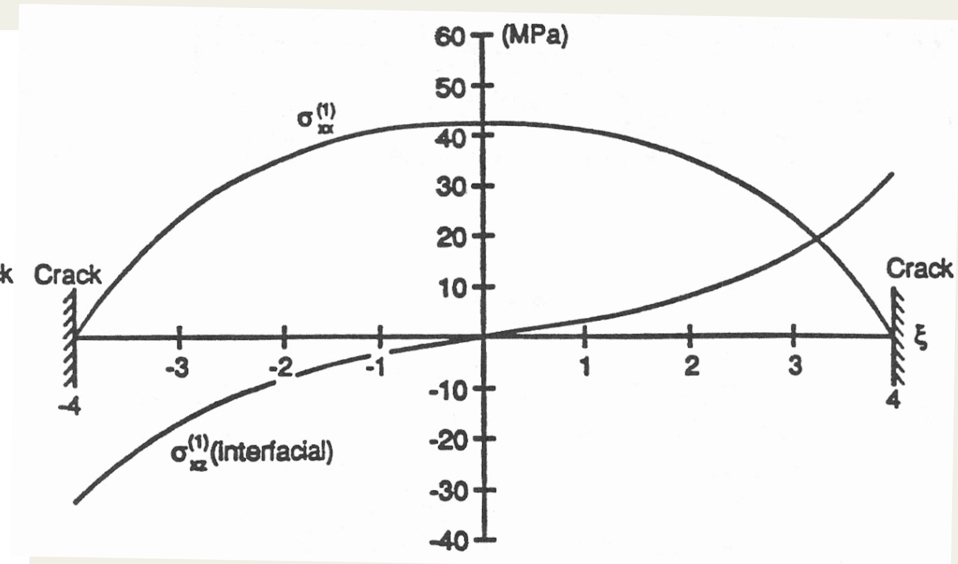
$$\rho = \frac{l}{t_{90}}$$

α_1 , α_2: functions of ply properties

Stress Distributions in 90-deg Plies



Hashin



Shear lag

Remarks:

- Hashin: Crack interactions, Shear lag: No crack interaction
- Hashin: Correct shear stress, Shear lag: Incorrect at crack planes
- Hashin: Thru-thickness stress, Shear lag: Not calculated

Average E-modulus of Cracked Cross Ply Laminate

Complementary Energies of Uncracked and Cracked Laminate (unit cell):

$$\Pi_0^* = \frac{1}{2}\frac{\sigma_c^2}{E_{x0}}.2Ah \qquad \Pi^* = \frac{1}{2}\frac{\sigma_c^2}{E_x}.2Ah$$

Principle of Minimum Complementary Energy:

$$\frac{1}{2}\frac{\sigma_c^2}{E_{x0}}.2Ah + \Pi' \geq \frac{1}{2}\frac{\sigma_c^2}{E_x}.2Ah$$

Π' : Change due to cracks

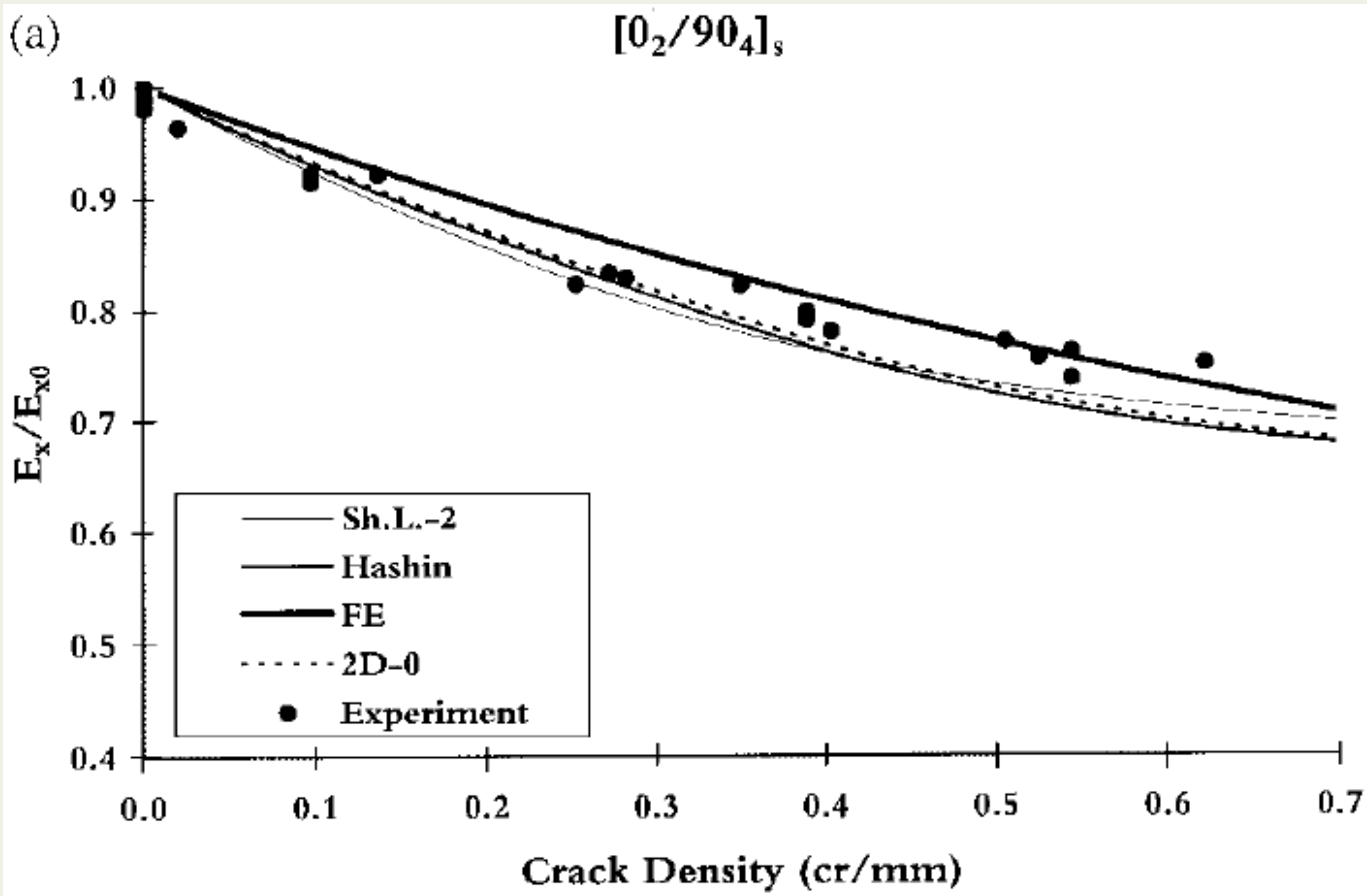
$$\Rightarrow \boxed{\frac{1}{E_x} \leq \frac{1}{E_{x0}} + \frac{\Pi'}{\sigma_c^2 Ah}.} \qquad \Pi' = 2\int_{-l}^{l}\int_0^{t_{90}} W_{90}\,\mathrm{d}z\,\mathrm{d}x + 2\int_{-l}^{l}\int_{t_{90}}^{h} W_0\,\mathrm{d}z\,\mathrm{d}x$$

Stress energy densities W_{90} and W_0 given by perturbation stresses

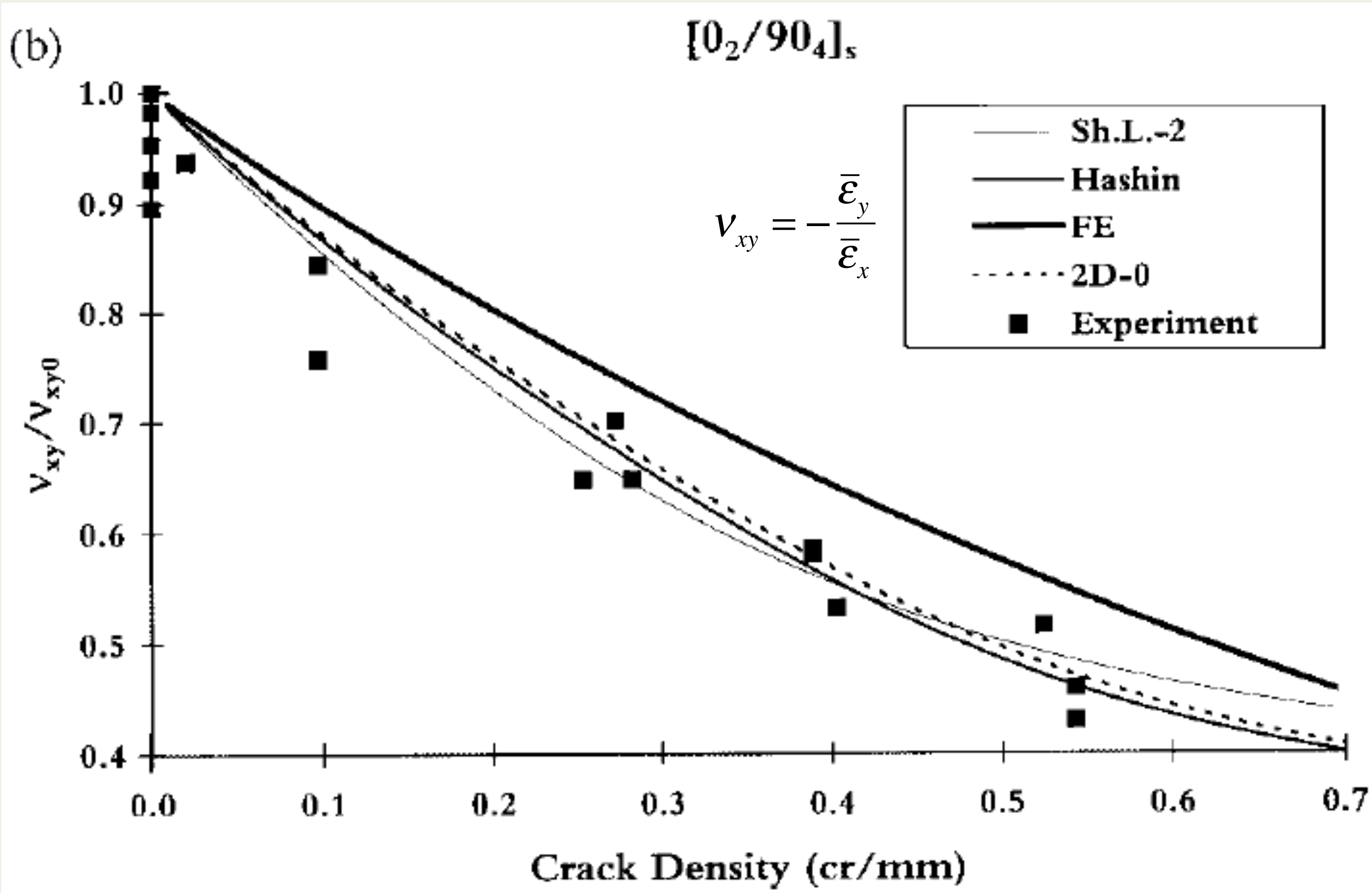
Note: E_x is lower bound!

Note: Average Poisson's ratio calculated from averaging strains over the unit cell.

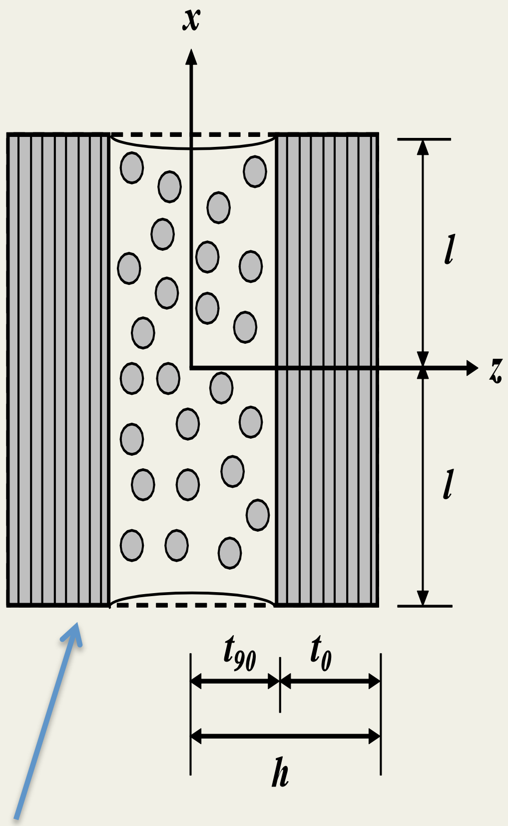
Average Axial E-modulus

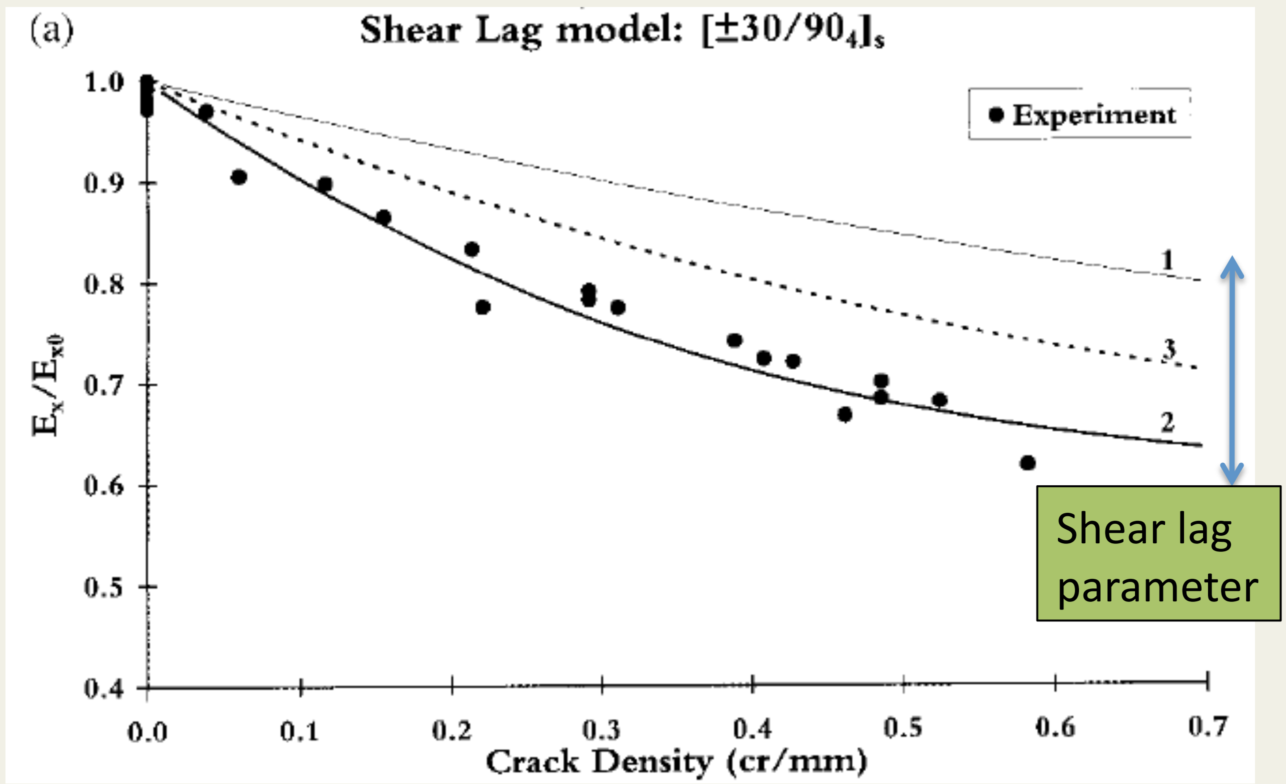


Average Axial Poisson's Ratio



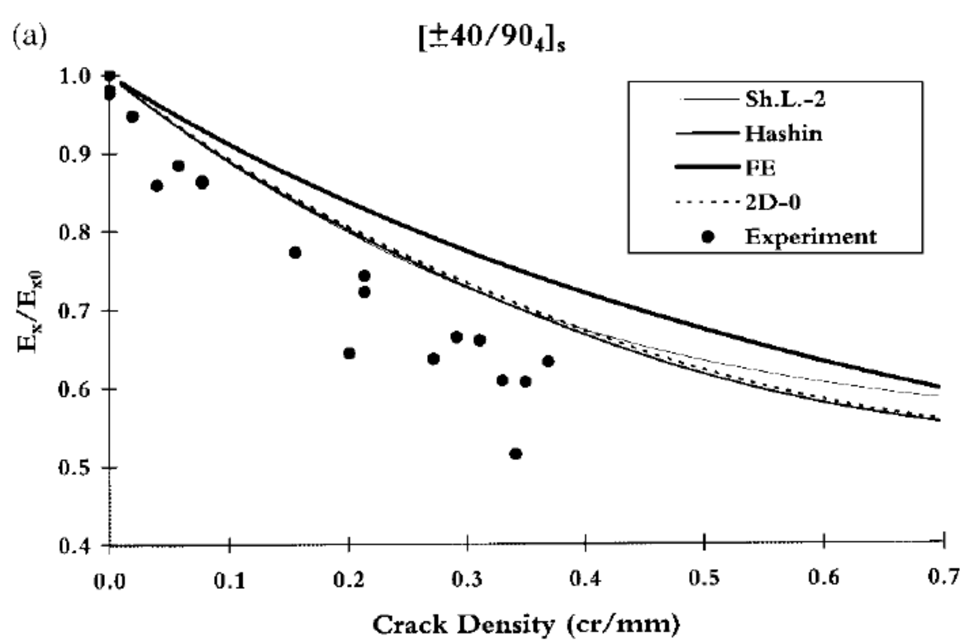
Treatment of $[\pm\theta/90]_s$ Laminates as Equivalent Cross Ply Laminates

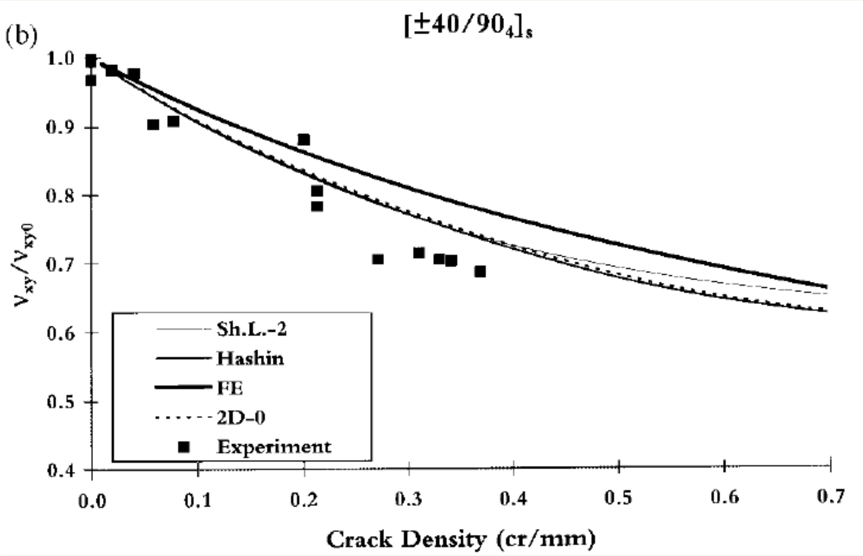




Replace 0-deg plies with averaged properties of $[\pm\theta]$ laminate in x-z coordinates

[±40/90]$_s$ Laminates



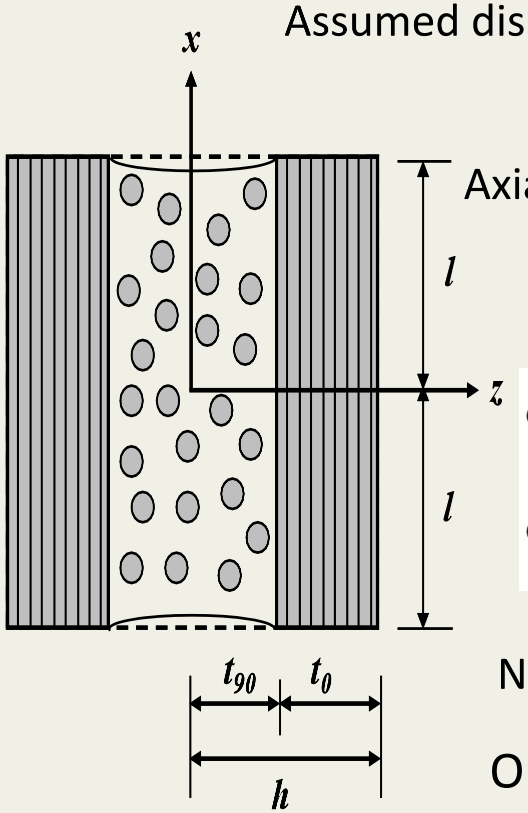


Estimates not always reliable.

Generalized Plain Strain Analysis - McCartney's Model

Assumed displacement field: $u_m = u_m(x,z),\ v_m = \varepsilon_T^c y, \qquad w_m = w_m(x,z)$

ε_T^c : Constant strain in the y-direction



Axial stresses after cracking assumed constant in z-direction (as in Hashin)

$$\sigma_{xx}^{0} = \sigma_{xx0}^{0} + C(x), \qquad \sigma_{xx}^{90} = \sigma_{xx0}^{90} - \lambda C(x),$$

Stress solution:

$$\sigma_{xz}^{0} = C'(x)\left[(1+\lambda)t_{90} - z\right], \qquad \sigma_{xz}^{90} = \lambda C'(x) z$$

$$\sigma_{zz}^{0} = \frac{1}{2} C''(x)\left[(1+\lambda)t_{90} - z\right]^2, \qquad \sigma_{zz}^{90} = \frac{1}{2} C''(x)\left[(1+\lambda)t_{90}^2 - z^2\right].$$

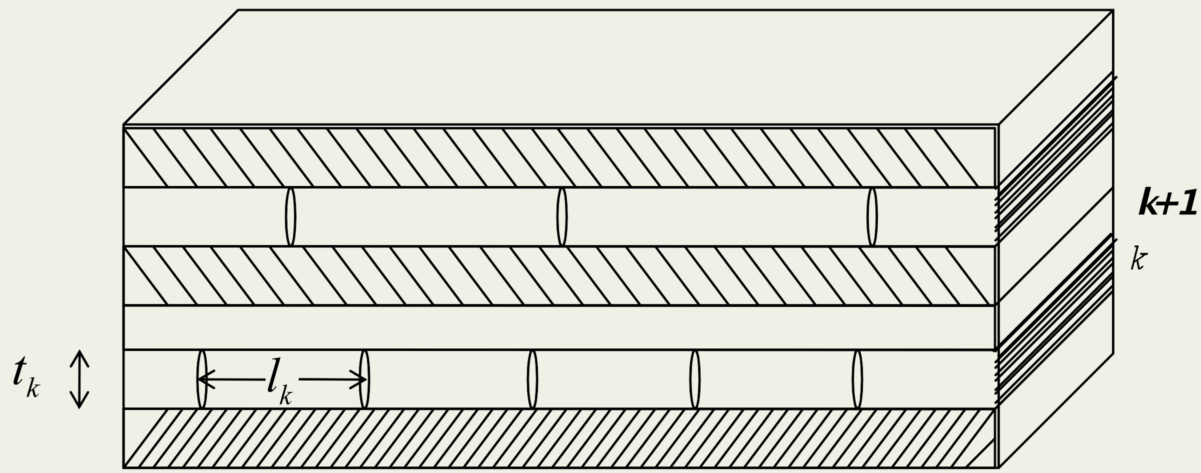
Note: Identical to Hashin if $C(x) = \frac{1}{\lambda}\sigma_{xx}^{90}\phi(x)$ $\qquad \lambda = \frac{t_0}{t_{90}}$

On using equilibrium and constitutive eqns: $FC^{IV}(x) - GC''(x) + HC(x) = 0$

F, G and H: functions of ply properties

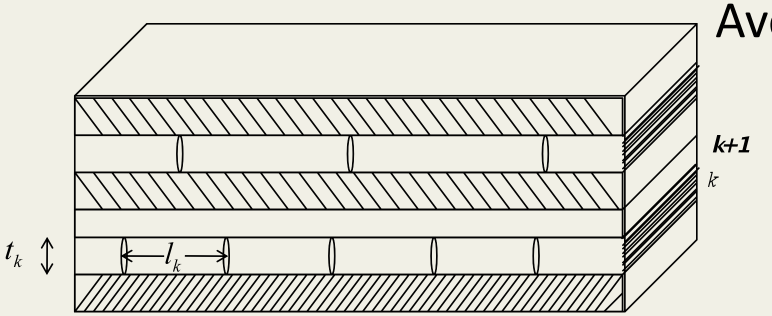
C(x) determined by using boundary conditions

Methods Based on Crack Surface Displacements



<u>Motivation</u>:
Consider a representative volume of a general laminate containing ply cracks in multiple orientations. Stress field in this laminate cannot be solved analytically. If the crack surface displacements can be estimated, then the overall (average) moduli can be estimated from the <u>additional strains</u> in terms of these displacements.

Gudmundson-Zang Model (1993)



RVE of volume V

Average stresses and strains in <u>uncracked</u> laminate:

$$\bar{\sigma}_{ij} = \sum_{k=1}^{N} V_k \sigma_{ij}^k, \quad \bar{\varepsilon}_{ij} = \sum_{k=1}^{N} V_k \varepsilon_{ij}^k$$

V_k: volume fraction of kth ply $\quad \sum_{k=1}^{N} V_k = 1$

<u>After cracking</u>, average stresses: $\bar{\sigma}_{ij}^{(a)} = \sum_{k=1}^{N} V_k \sigma_{ij}^{k(a)}$

Effective laminate strains: $\bar{\varepsilon}_{ij}^{(e)} = \frac{1}{2V} \int_{\Gamma^{out}} \left(u_i n_j + u_j n_i\right) \mathrm{d}\Gamma$

u_i: displacement

normal on outer surface Γ^{out}

Effective lamina strains: $\varepsilon_{ij}^{k(e)} = \frac{1}{2V^k} \int_{\Gamma^{kout}} \left(u_i^k n_j^k + u_j^k n_i^k\right) \mathrm{d}\Gamma \;\rightarrow\; \bar{\varepsilon}_{ij}^{(e)} = \sum_{k=1}^{N} V_k \varepsilon_{ij}^{k(e)}$

Using divergence theorem, obtain: $\boxed{\varepsilon_{ij}^{k(e)} = \varepsilon_{ij}^{k(a)} + \Delta\varepsilon_{ij}^{k}.}$

→ Effective strains = average strains + additional strains from cracks

Additional Strains From Crack Surface Displacements

Assumptions in the Gudmundson-Zang Model:

- The surface displacements of a ply crack in a finite-thickness laminate are equal to those of a crack in an infinite, homogeneous transversely isotropic medium.
- There is no effect of orientation of a cracked ply.
- There is no coupling between crack opening displacements of different plies.

ERRORS DUE TO THESE ASSUMPTIONS CANNOT BE ASSESSED INDEPENDENTLY

Lundmark-Varna Model (2005)

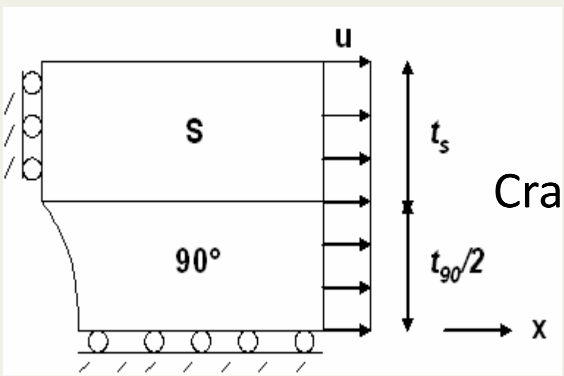
As in Gudmundson-Zang model, $\{\bar{\varepsilon}_{ij}\}_k^a = \{\bar{\varepsilon}_{ij}\}^{LAM} + \{\bar{\beta}_{ij}\}_k$

i.e., overall (effective) strain after cracking
= strain before cracking (given by laminate theory) + strain from crack surface displ.

$$\{\bar{\beta}_{ij}\}_k = \frac{1}{2V^k}\int_{\Gamma^{kc}}\left(u_i^k n_j^k + u_j^k n_i^k\right)d\Gamma$$

expressed for a general case in terms of ply properties, ply orientation, crack spacing and crack surface displacements.

Example:



Crack opening displacement: $u_{2an} = A + B\left(\frac{E_2}{E_x^s}\right)^n$

E_2: transverse ply modulus
E^s: axial modulus of sublaminate S

A, B, and *n* obtained by FE parametric study

Other Developments in Micro-Damage Mechanics

- Numerous variations of shear lag method proposed, including some who claim “2-D”
- Improvements in Hashin’s variational analysis made by relaxing his simplifying assumptions (Varna & Berglund, 1991-94)
- Self-consistent approximation to effective properties also proposed (for cross ply laminates)
- For inclined cracks in one orientation, Li (1999, 2001) proposed a semi-analytical method

Concluding Remarks on Micro-Damage Mechanics

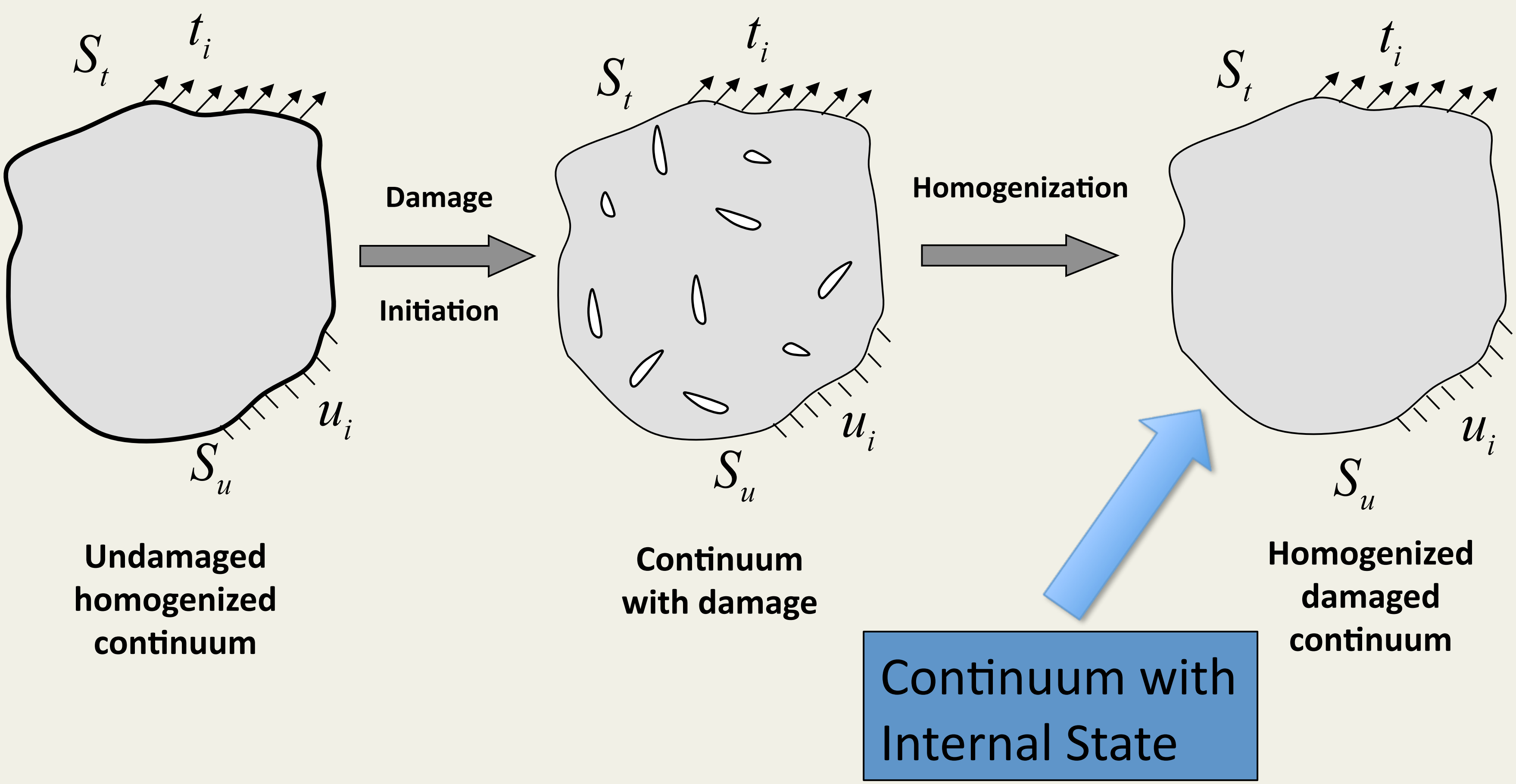
- Limited by our ability to find stress field
- Necessary for analysis of evolution of progressive cracking
- So far has only been possible to do micro-mesoscale, i.e., to describe mesoscale constitutive behavior from analysis at microscale, and only for very limited cases (mostly cross ply laminates)
- A “blind” multiscale analysis (without knowledge of damage) will be a futile exercise (perhaps self-deceiving)

PART 3:
MACRO-DAMAGE MECHANICS

Desirable Properties of a Damage Modeling Framework

- Should be based on physical mechanisms
- Should have wide applicability (not limited to one or two cases)
- Should be applicable to structural analysis

Constitutive Description of Anisotropic, Elastic Solid with Damage



Early Concepts of Internal State

Kachanov (1958) concept of material degradation (in metal creep) given by a "continuity" variable $\phi = 1$ (virgin state), $\phi = 0$ (failure)

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -A\left(\frac{\sigma}{\phi}\right)^m$$

A, m : material constants; σ : max principal tensile stress

Define damage variable (Robotnov, 1969) $\omega = 1 - \phi$

Effective stress: $\tilde{\sigma} = \dfrac{\sigma}{1-\omega}$.

Hooke's Law: $\sigma = \tilde{E}\varepsilon_e, \tilde{\sigma} = E\varepsilon_e,$

$$\tilde{\sigma} = \frac{\sigma}{1-\omega} = \frac{E}{\tilde{E}}\sigma \Rightarrow \omega = 1 - \frac{\tilde{E}}{E} .$$

Effective Elastic Constant: $\tilde{E} = (1-\omega)E$

3-D Generalization (Chaboche, 1984):

Effective stress: $\boldsymbol{\sigma}^* = (\mathbf{I} - \mathbf{D})^{-1} : \boldsymbol{\sigma}$.

Effective Elasticity Tensor: $\tilde{\mathbf{E}} = (\mathbf{I} - \mathbf{D}) : \mathbf{E}$

Fourth Order Damage Tensor: $\mathbf{D} = \mathbf{I} - \mathbf{E} \bullet \mathbf{E}^{-1}$

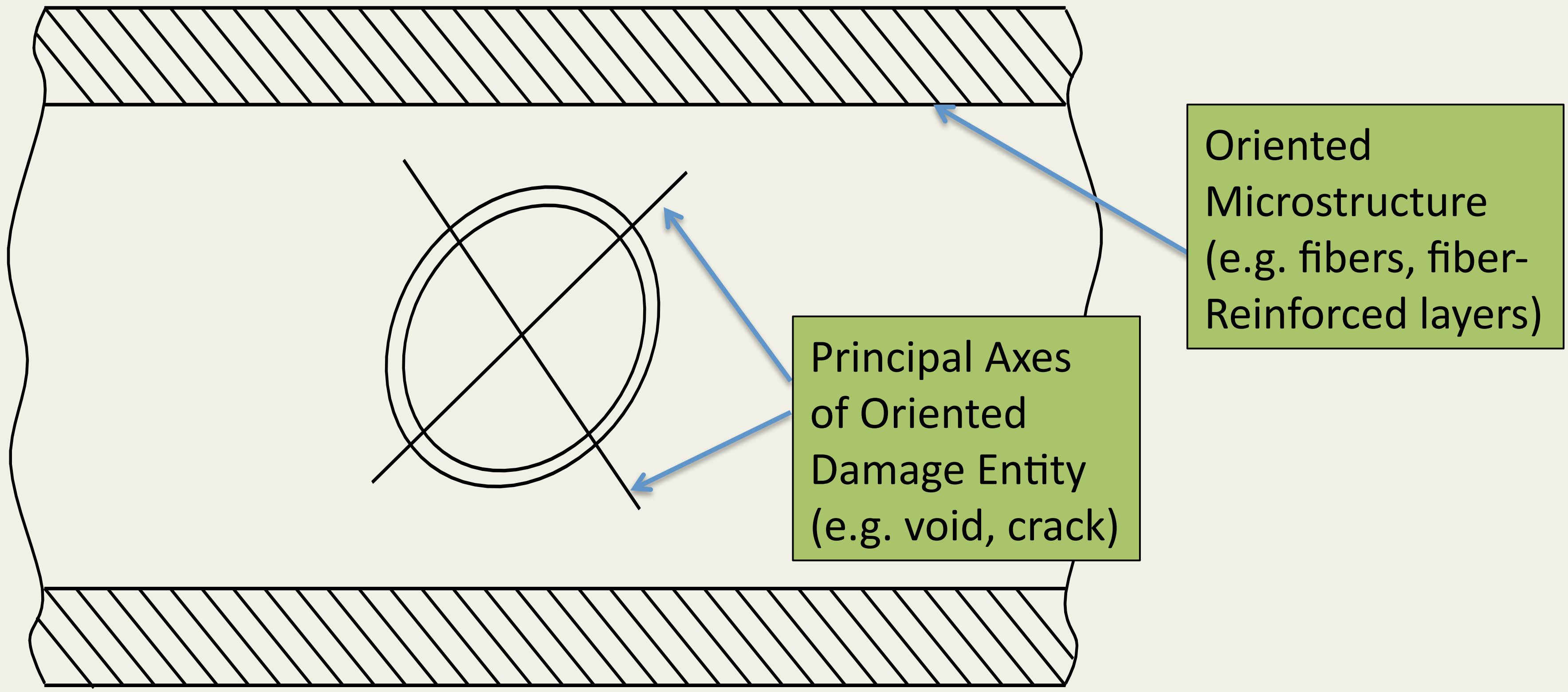
Remarks on Kachanov Concept of Internal State Characterization and Its 3-D Generalization

- Although a pioneering concept, Kachanov's "discontinuity" was not based on observed microstructure (not possible in 1958)
- A formal 3-D generalization (e.g. Chaboche) is therefore not on solid physical ground
- Today's techniques can reveal specific details of microstructure (internal state) responsible for permanent changes in material response, and should therefore be basis for characterization

Observations Before Launching a Damage Characterization

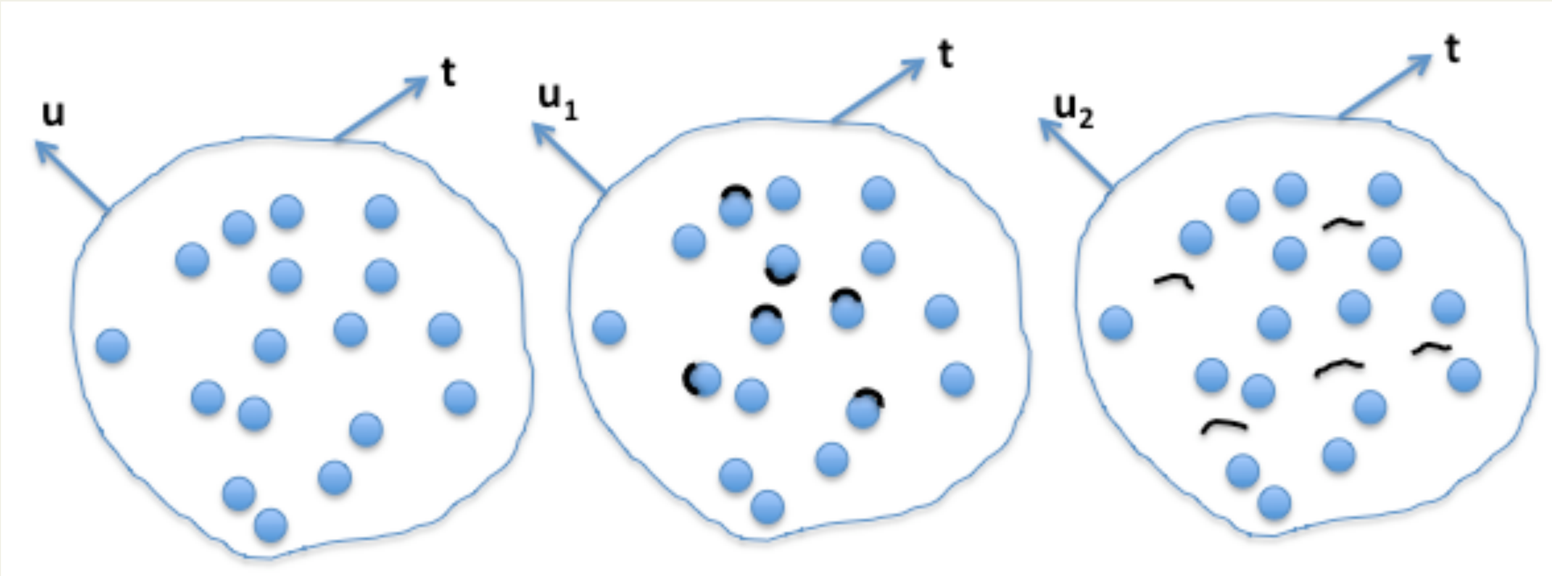
- Not all microstructure is “damage”, only those changes (rearrangements) that induce permanent response changes
- Microstructural entities can stay unchanged while producing other entities (cracks) that induce permanent response changes
- Damage characterization should be *motivated by the need, based on “knowledge”, and not “over informed” by microstructure*

Further Considerations Before Damage Characterization



Oriented microstructure induces anisotropy,
while oriented damage induces ***<u>changes</u>*** in this anisotropy

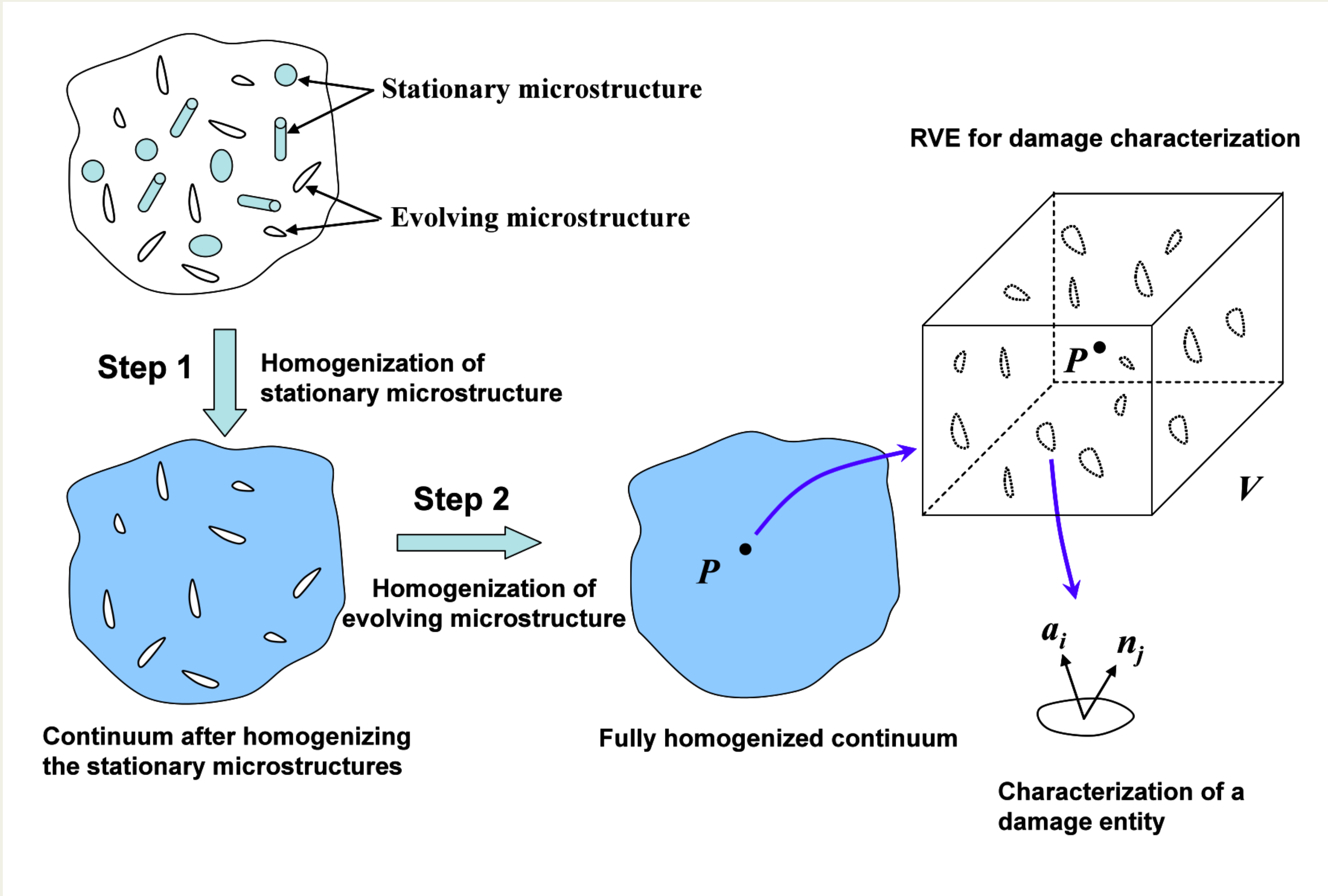
Microstructure vs. Damage



Pristine state Damage type 1 Damage type 2

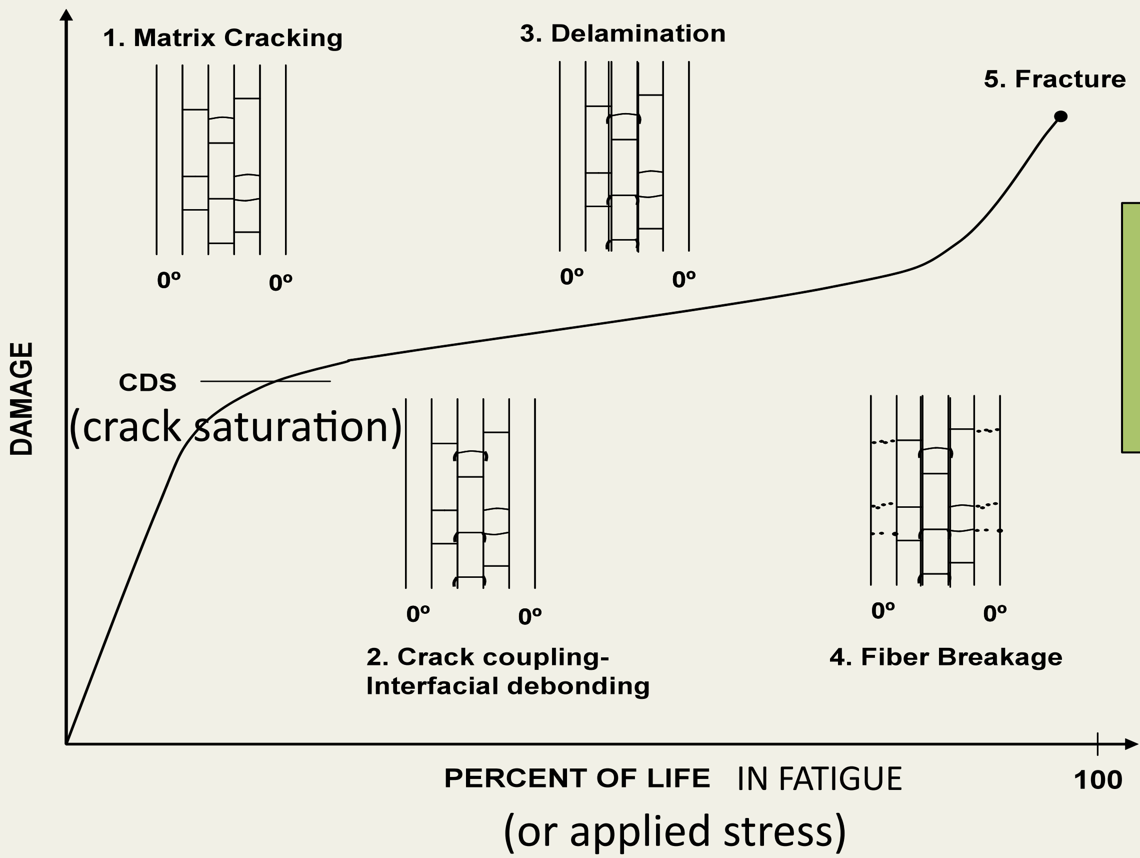
Characteristic length scales of damage not always the same as those of microstructure

Proposed Characterization of Damage in Composite Materials



Note: RVE of damage NOT the same as that of microstructure

Selection of Damage for Characterization

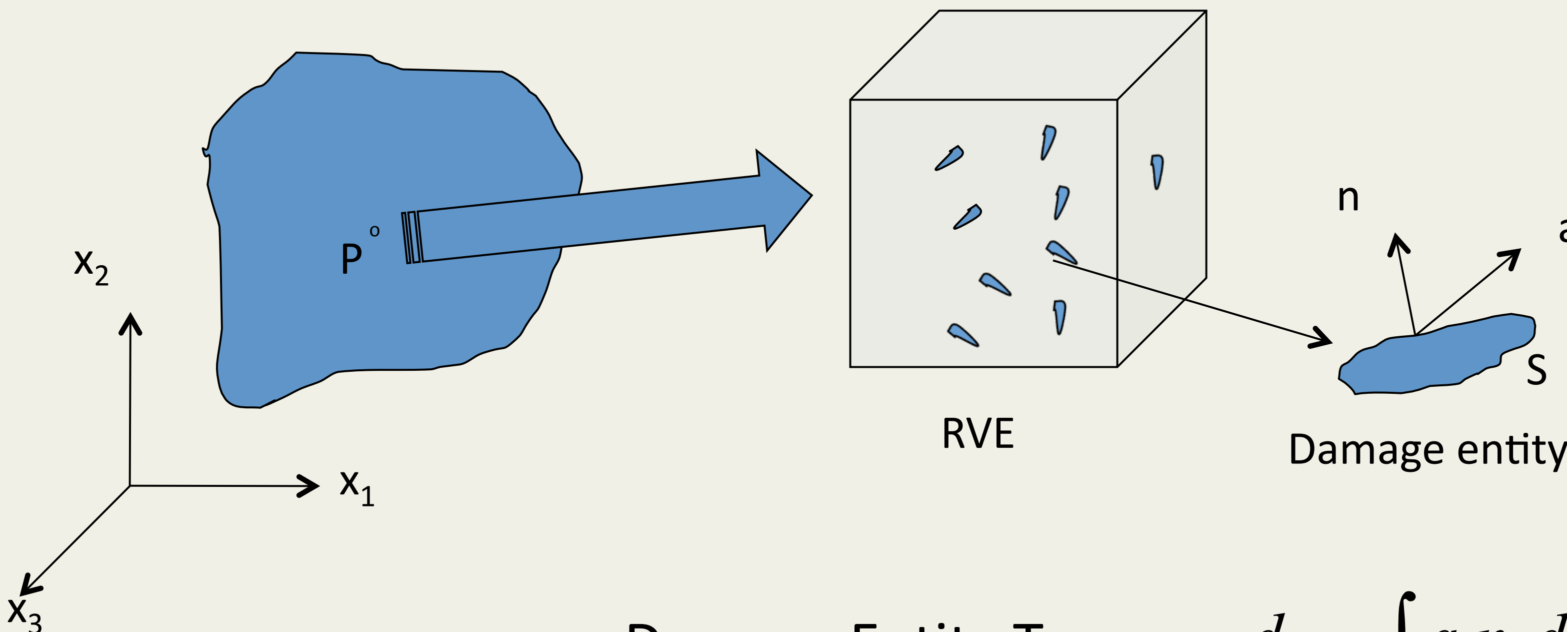


"Damage" defined here as distributed (multiple) cracking

Isolated single cracks and heavily coupled localized cracking not included

Purpose: Description of meso-scale (RVE-averaged) properties

A Tensor Representation of Damage

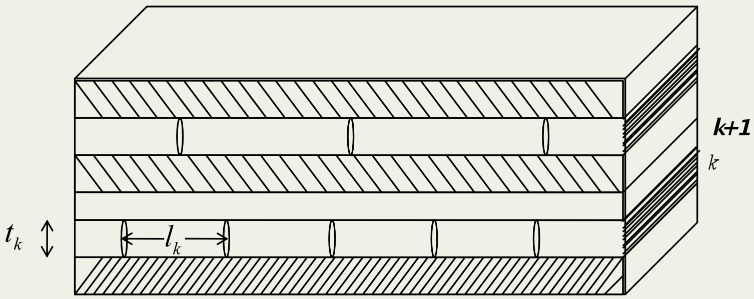


Damage Entity Tensor: $d_{ij} = \int_S a_i n_j dS$

(Vakulenko-Kachanov, 1971)

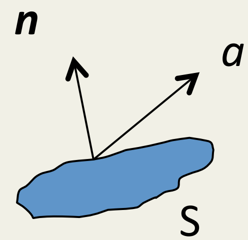
Damage vector **a** represents any selected INFLUENCE of damage entity

Damage Mode Tensors



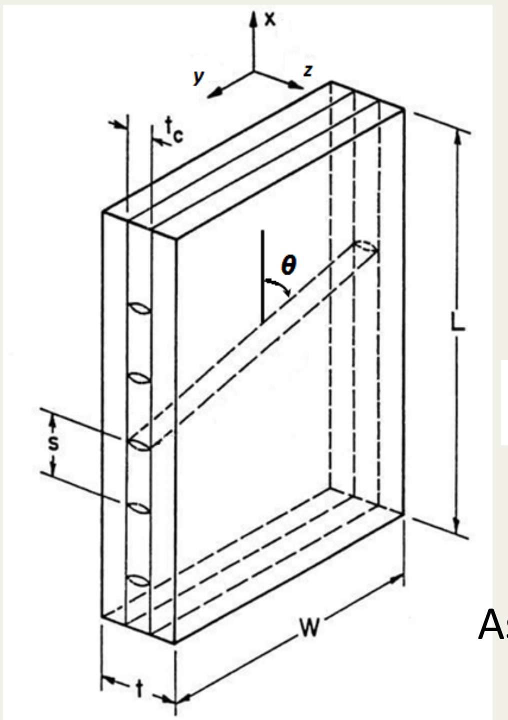
RVE of volume V

$$d_{ij} = \int_S a_i n_j \mathrm{d}S$$



$$D_{ij}^{(\alpha)} = \frac{1}{V}\sum_{k_\alpha}\left(d_{ij}\right)_{k_\alpha}$$

where k_α is the number of damage entities in the α^{th} mode



$$a_i = an_i + bm_i \qquad n_i m_i = 0$$

a: crack opening displacement

b: crack sliding displacement

$$d_{ij} = d_{ij}^1 + d_{ij}^2$$

$$d_{ij}^1 = \int_S a n_i n_j \mathrm{d}S, \qquad d_{ij}^2 = \int_S b m_i n_j \mathrm{d}S$$

$$D_{ij}^{(\alpha)} = D_{ij}^{1(\alpha)} + D_{ij}^{2(\alpha)}$$

$$D_{ij}^{1(\alpha)} = \frac{1}{V}\sum_{k_\alpha}\left(d_{ij}^1\right)_{k_\alpha}, \quad D_{ij}^{2(\alpha)} = \frac{1}{V}\sum_{k_\alpha}\left(d_{ij}^2\right)_{k_\alpha}.$$

Assume *b* = 0

$$\boxed{D_{ij}^{(\alpha)} = D_{ij}^{1(\alpha)} = \frac{1}{V}\sum_{k_\alpha}\left[\int_S a n_i n_j \mathrm{d}S\right]_{k_\alpha}.}$$

Thermodynamics Framework for Materials Response (CDM)

- Classical framework of thermodynamics with internal variables applied to homogeneous, anisotropic composite materials containing distributed multiple cracks
- Damage mode tensors used as internal variables
- Small elastic strains used (adequate for composite laminates of e.g. glass/epoxy and carbon/epoxy that are most widely used)
- Extensions to more general cases possible

Continuum Damage Mechanics - Elastic

Thermodynamic State

- Strain, ε_{ij}
- Temperature, T
- Temperature gradient, $T_{,i}$
- Damage, D_{ij}

Response Functions

- Stress, σ_{ij} ; Heat flux, q_i
- Specific Helmholtz free energy, ψ
- Specific entropy, η
- Damage rate, $\dot{D}_{ij}$

Truesdell's Equipresence Principle

Clausius-Duhem Inequality

Isothermal Case

$$\psi(\varepsilon_{ij}, D_{ij}) \qquad \sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}}$$

Material symmetry - irreducible integrity bases

Materials Response Functions

Truesdell's Principle of Equipresence: Response functions should depend on all variables of thermodynamic state unless independence required by physical laws

$$\sigma_{ij} = \sigma_{ij}\left(\varepsilon_{kl}, T, g_k, D_{kl}^{(\alpha)}\right)$$

$$\psi = \psi\left(\varepsilon_{kl}, T, g_k, D_{kl}^{(\alpha)}\right)$$

$$\eta = \eta\left(\varepsilon_{kl}, T, g_k, D_{kl}^{(\alpha)}\right) \qquad g_i = T_{,i}$$

$$q = q\left(\varepsilon_{kl}, T, g_k, D_{kl}^{(\alpha)}\right)$$

$$\dot{D}_{kl}^{(\alpha)} = \dot{D}_{kl}^{(\alpha)}\left(\varepsilon_{kl}, T, g_k, D_{kl}^{(\beta)}\right).$$

Clausius-Duhem inequality:

$$\sigma_{ij}\dot{\varepsilon}_{ij} - \rho\dot{\psi} - \rho\dot{T}\eta - \frac{q_i g_i}{T} \geq 0$$

$$\psi = u - T\eta$$

u: specific internal energy per unit mass

$$\dot{\psi} = \frac{\partial \psi}{\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl} + \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial g_i}\dot{g}_i + \frac{\partial \psi}{\partial D_{kl}^{(\alpha)}}\dot{D}_{kl}^{(\alpha)}.$$

$$\left(\sigma_{ij} - \rho\frac{\partial \psi}{\partial \varepsilon_{kl}}\right)\dot{\varepsilon}_{ij} - \rho\left(\eta + \frac{\partial \psi}{\partial T}\right)\dot{T} - \rho\frac{\partial \psi}{\partial g_i}\dot{g}_i - \rho\sum_{\alpha}\frac{\partial \psi}{\partial D_{kl}^{(\alpha)}}\dot{D}_{kl}^{(\alpha)} - \frac{q_i g_i}{T} \geq 0$$

$$\Rightarrow \quad \sigma_{ij} = \rho\frac{\partial \psi}{\partial \varepsilon_{kl}} \qquad \eta = -\frac{\partial \psi}{\partial T} \qquad \frac{\partial \psi}{\partial g_i} = 0\,.$$

Reduced Response Functions

$$\sigma_{ij} = \sigma_{ij}\left(\varepsilon_{kl}, T, D_{kl}^{(\alpha)}\right)$$
$$\psi = \psi\left(\varepsilon_{kl}, T, D_{kl}^{(\alpha)}\right)$$
$$\eta = \eta\left(\varepsilon_{kl}, T, D_{kl}^{(\alpha)}\right)$$
$$q = q\left(\varepsilon_{kl}, T, g_k, D_{kl}^{(\alpha)}\right)$$
$$\dot{D}_{kl}^{(\alpha)} = \dot{D}_{kl}^{(\alpha)}\left(\varepsilon_{kl}, T, g_k, D_{kl}^{(\beta)}\right).$$

Internal dissipation inequality:

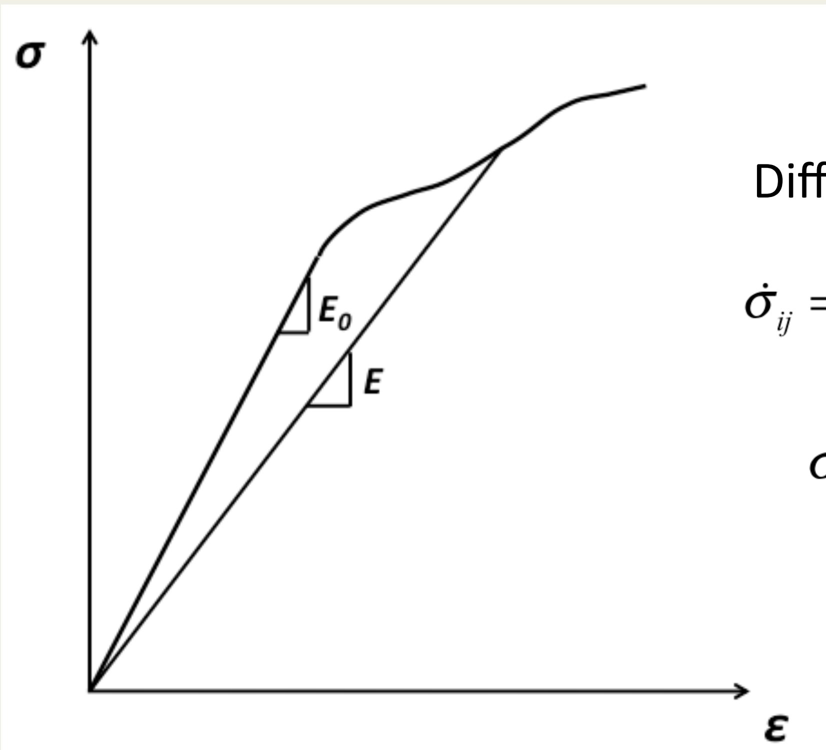
$$\sum_{\alpha} R_{kl}^{(\alpha)} \dot{D}_{kl}^{(\alpha)} - \frac{q_i g_i}{T} \geq 0 \qquad R_{kl}^{(\alpha)} = -\rho \frac{\partial \psi}{\partial D_{kl}^{(\alpha)}} \, .$$

For isothermal conditions (T = 0, g_i = 0):

$$\sigma_{ij} = \sigma_{ij}\left(\varepsilon_{kl}, D_{kl}^{(\alpha)}\right)$$
$$\psi = \psi\left(\varepsilon_{kl}, D_{kl}^{(\alpha)}\right)$$
$$\dot{D}_{kl}^{(\alpha)} = \dot{D}_{kl}^{(\alpha)}\left(\varepsilon_{kl}, g_k, D_{kl}^{(\beta)}\right)$$

and $$\sum_{\alpha} R_{kl}^{(\alpha)} \dot{D}_{kl}^{(\alpha)} \geq 0 \, .$$

Stress-Strain Response at Fixed Damage



$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{kl}}$$

Differentiating,

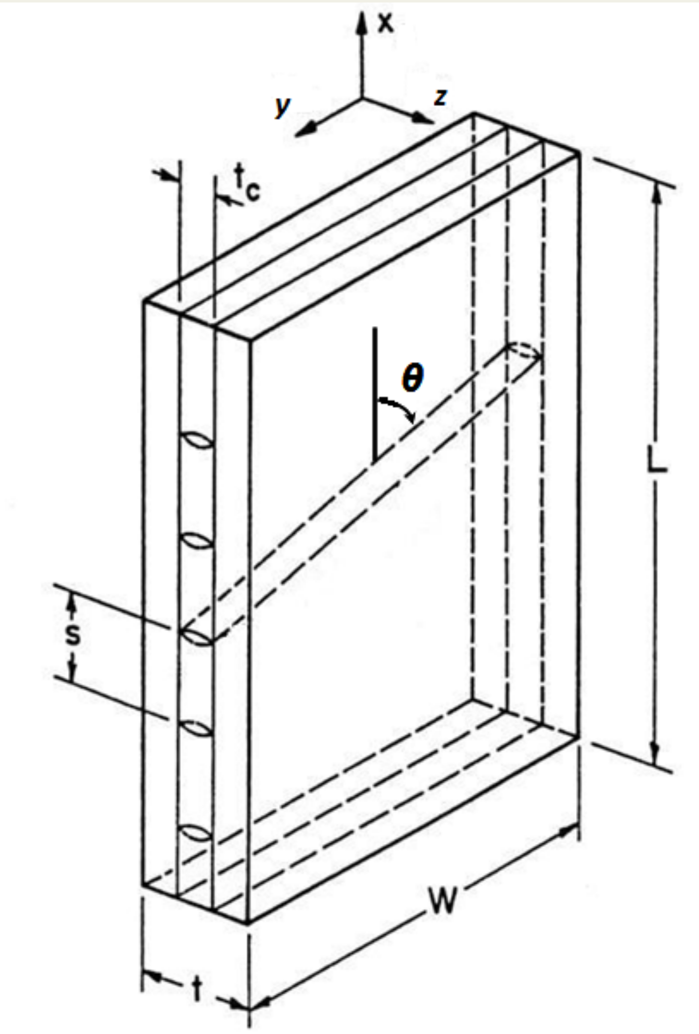
$$\dot{\sigma}_{ij} = \rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \rho \sum_{\alpha} \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial D^{(\alpha)}_{mn}} \dot{D}^{(\alpha)}_{mn} \,.$$

$$\dot{\sigma}_{ij} = \rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} - \sum_{\alpha} \frac{\partial R^{(\alpha)}_{mn}}{\partial \varepsilon_{ij}} \dot{D}^{(\alpha)}_{mn} \,.$$

Or $$\dot{\sigma}_{ij} = C_{ij\,kl} \dot{\varepsilon}_{kl} - \sum_{\alpha} K^{(\alpha)}_{ij\,mn} \dot{D}^{(\alpha)}_{mn}$$

where $$C_{ij\,kl} = \rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \qquad K^{(\alpha)}_{ij\,mn} = \frac{\partial R^{(\alpha)}_{mn}}{\partial \varepsilon_{ij}} = -\rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial D^{(\alpha)}_{mn}} \,.$$

Damage Tensor Components (One Damage Mode)



$$D_{ij}^{(\alpha)} = D_{ij}^{1(\alpha)} = \frac{1}{V}\sum_{k_\alpha}\left[\int_S a n_i n_j \,\mathrm{d}S\right]_{k_\alpha} \quad .$$

$$V = L.W.t$$

$$S = \frac{W.t_c}{\sin\theta}$$

$$a = \kappa t_c$$

$$n_i = (\sin\theta, \quad \cos\theta, \quad 0)$$

$$\boxed{D_{ij}^{(\alpha)} = \frac{\kappa t_c^2}{st\sin\theta} n_i n_j}$$

$$\alpha = 1$$

(one damage mode)

κ (kappa): Constraint parameter

Helmholtz Free Energy (Polynomial Expansion)

Let $\psi = \psi_P\left(\varepsilon_{ij}, D_{ij}^{(\alpha)}\right)$ be a polynomial function

Expansion in terms of irreducible integrity bases (polynomial invariants)

For orthotropic symmetry, one damage mode: (Adkins, 1960)

$$
\begin{aligned}
&\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}^2, \varepsilon_{31}^2, \varepsilon_{12}^2, \varepsilon_{23}\varepsilon_{31}\varepsilon_{12} \\
&D_{11}, D_{22}, D_{33}, D_{23}^2, D_{31}^2, D_{12}^2, D_{23}D_{31}D_{12} \\
&\varepsilon_{23}D_{23}, \varepsilon_{31}D_{31}, \varepsilon_{12}D_{12}, \\
&\varepsilon_{31}\varepsilon_{12}D_{23}, \varepsilon_{12}\varepsilon_{23}D_{31}, \varepsilon_{23}\varepsilon_{31}D_{12}, \\
&\varepsilon_{23}D_{31}D_{12}, \varepsilon_{31}D_{12}D_{23}, \varepsilon_{12}D_{23}D_{31}\,.
\end{aligned}
$$

Thin laminates

One intralaminar cracking mode

$$
\begin{aligned}
&\varepsilon_1, \varepsilon_2, \varepsilon_6^2 \\
&D_1, D_2, D_6^2 \\
&\varepsilon_6 D_6
\end{aligned}
$$

$$\varepsilon_1 \equiv \varepsilon_{11}, \varepsilon_2 \equiv \varepsilon_{22}, \varepsilon_6 \equiv \varepsilon_{12}, D_1 \equiv D_{11}, D_2 \equiv D_{22}, D_6 \equiv D_{12}$$

$$
\begin{aligned}
\rho\psi = P_0 &+ \left\{c_1\varepsilon_1^2 + c_2\varepsilon_2^2 + c_3\varepsilon_6^2 + c_4\varepsilon_1\varepsilon_2\right\} \\
&+ \left\{c_5\varepsilon_1^2 D_1 + c_6\varepsilon_1^2 D_2\right\} + \left\{c_7\varepsilon_2^2 D_1 + c_8\varepsilon_2^2 D_2\right\} + \left\{c_9\varepsilon_6^2 D_1 + c_{10}\varepsilon_6^2 D_2\right\} \\
&+ \left\{c_{11}\varepsilon_1\varepsilon_2 D_1 + c_{12}\varepsilon_1\varepsilon_2 D_2\right\} + \left\{c_{13}\varepsilon_1\varepsilon_6 D_6 + c_{14}\varepsilon_2\varepsilon_6 D_6\right\} \\
&+ P_1(\varepsilon_p, D_q) + P_2(D_q)
\end{aligned}
$$

Stiffness-Damage Relationships

$$\sigma_p = \rho \frac{\partial \psi}{\partial \varepsilon_p}$$

p = 1, 2, 6

$$\mathrm{d}\sigma_p = \rho \frac{\partial \psi}{\partial \varepsilon_p \partial \varepsilon_q} \mathrm{d}\varepsilon_q + \rho \frac{\partial \psi}{\partial \varepsilon_p \partial D_r} \mathrm{d}D_r = C_{pq} \mathrm{d}\varepsilon_q + K_{pr} \mathrm{d}D_r$$

$$C_{pq} = \rho \frac{\partial \psi}{\partial \varepsilon_p \partial \varepsilon_q}$$

$$\mathbf{C_{pq}} = \mathbf{C^0_{pq}} + \mathbf{C^{(1)}_{pq}}$$

$$\mathbf{C^0_{pq}} = \begin{bmatrix} 2c_1 & c_4 & 0 \\ & 2c_2 & 0 \\ Symm & & 2c_3 \end{bmatrix} = \begin{bmatrix} \dfrac{E^0_x}{1-\nu^0_{xy}\nu^0_{yx}} & \dfrac{\nu^0_{xy}E^0_y}{1-\nu^0_{xy}\nu^0_{yx}} & 0 \\ & \dfrac{E^0_y}{1-\nu^0_{xy}\nu^0_{yx}} & 0 \\ Symm & & G^0_{xy} \end{bmatrix}$$

$$\mathbf{C^{(1)}_{pq}} = \begin{bmatrix} 2c_5D_1 + 2c_6D_2 & c_{11}D_1 + c_{12}D_2 & c_{13}D_6 \\ & 2c_7D_1 + 2c_8D_2 & c_{14}D_6 \\ Symm & & 2c_9D_1 + 2c_{10}D_2 \end{bmatrix}$$

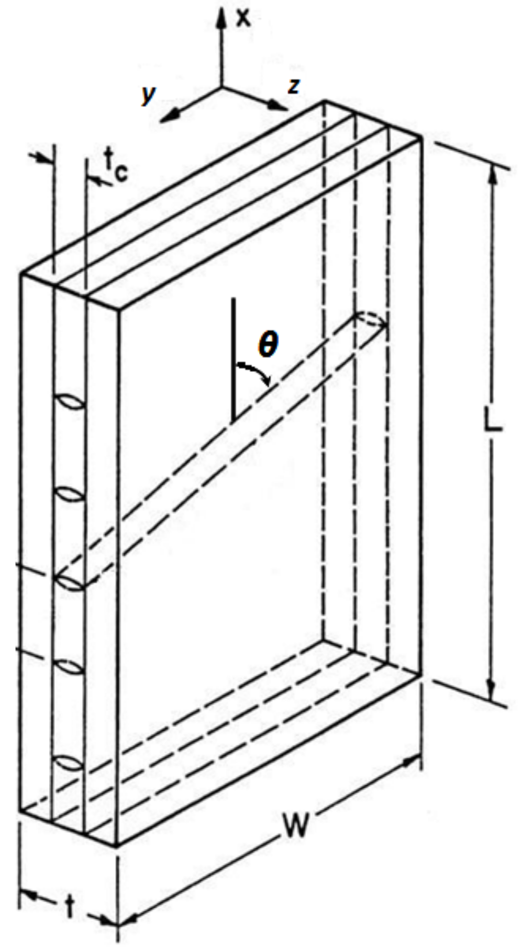
Constants c_i are multiplying terms in polynomial expansion

Example: One set of inclined cracks

$$D_1 = D_{11}^{(1)} = \frac{\kappa t_c^2}{st} \sin\theta ,$$

$$D_2 = D_{22}^{(1)} = \frac{\kappa t_c^2}{st} \frac{\cos^2\theta}{\sin\theta} ,$$

$$D_6 = D_{12}^{(1)} = \frac{\kappa t_c^2}{st} \cos\theta \ .$$



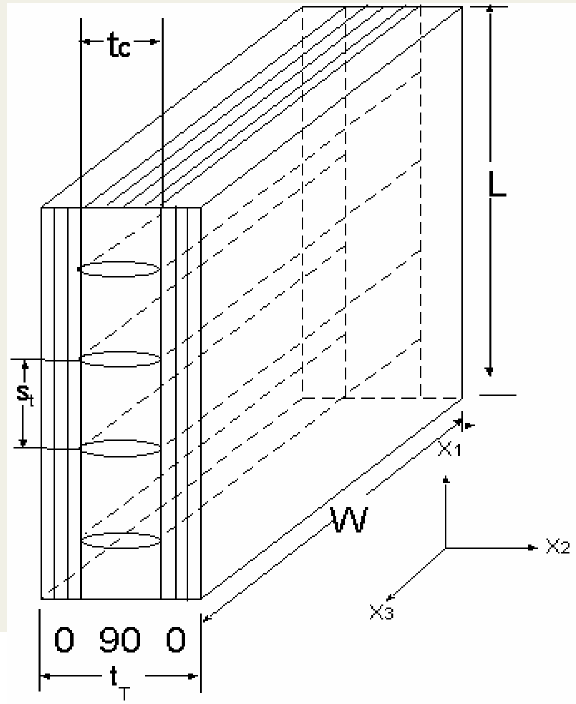
$$\mathbf{C_{pq}} = \begin{bmatrix} \dfrac{E_x^0}{1-\nu_{xy}^0\nu_{yx}^0} & \dfrac{\nu_{xy}^0 E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} & 0 \\ & \dfrac{E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} & 0 \\ Symm & & G_{xy}^0 \end{bmatrix}$$

$$+\frac{\kappa t_c^2}{st}\sin\theta \begin{bmatrix} 2c_5 + 2c_6\cot^2\theta & c_{11} + c_{12}\cot^2\theta & c_{13}\cot\theta \\ & 2c_7 + 2c_8\cot^2\theta & c_{14}\cot\theta \\ Symm & & 2c_9 + 2c_{10}\cot^2\theta \end{bmatrix} .$$

Cross Ply Laminate with Transverse Cracks

$\theta = 90^{\circ}$

$$D_{11} = \frac{\kappa t_c{}^2}{s_1 t_T}$$



$$\mathbf{C_{pq}} = \begin{bmatrix} \dfrac{E_x^0}{1-\nu_{xy}^0\nu_{yx}^0} & \dfrac{\nu_{xy}^0 E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} & 0 \\ & \dfrac{E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} & 0 \\ Symm & & G_{xy}^0 \end{bmatrix} + \frac{\kappa t_c^2}{st}\begin{bmatrix} 2a_1 & a_4 & 0 \\ & 2a_2 & 0 \\ Symm & & 2a_3 \end{bmatrix}$$

$a_1 = c_5, a_2 = c_7, a_3 = c_9$, and $a_4 = c_{11}$

Engineering Moduli for Cross Ply Laminates with Transverse Cracks

$$E_x = \frac{C_{11}C_{22} - C_{12}^2}{C_{22}} \qquad E_y = \frac{C_{11}C_{22} - C_{12}^2}{C_{11}}$$

$$\nu_{xy} = \frac{C_{11}}{C_{22}} \qquad G_{xy} = C_{66}$$

$$E_x = \frac{E_x^0}{1-\nu_{xy}^0\nu_{yx}^0} + 2\frac{\kappa t_c^2}{st}a_1 - \frac{\left[\frac{\nu_{xy}^0 E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} + \frac{\kappa t_c^2}{st}a_4\right]^2}{\frac{E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} + 2\frac{\kappa t_c^2 \sin\theta}{st}a_2}$$

$$E_y = \frac{E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} + 2\frac{\kappa t_c^2}{st}a_2 - \frac{\left[\frac{\nu_{xy}^0 E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} + \frac{\kappa t_c^2}{st}a_4\right]^2}{\frac{E_x^0}{1-\nu_{xy}^0\nu_{yx}^0} + 2\frac{\kappa t_c^2}{st}a_1}$$

$$\nu_{xy} = \frac{\frac{\nu_{xy}^0 E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} + \frac{\kappa t_c^2}{st}a_4}{\frac{E_y^0}{1-\nu_{xy}^0\nu_{yx}^0} + 2\frac{\kappa t_c^2}{st}a_2}$$

$$G_{xy} = G_{xy}^0 + 2\frac{\kappa t_c^2}{st}a_3 \ .$$

Note: Four unknown constants

κa_1, κa_2, κa_3, κa_4

needed to evaluate the elastic moduli.

These can be obtained by solving the equations for one state of damage (crack spacing s)

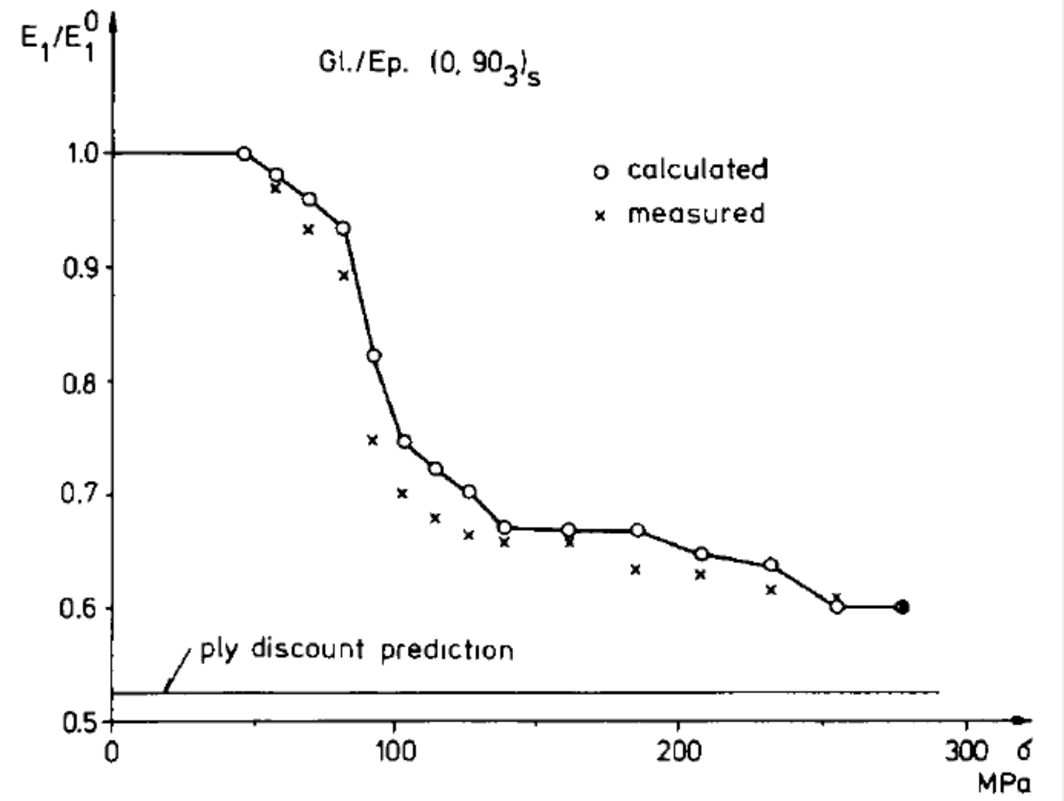
Prediction of Elastic Moduli for Cracked Cross Ply Laminates

$$a_1 = \frac{s_1 t}{2\kappa t_c^2}\left[\frac{E_x}{1-\nu_{xy}\nu_{yx}} - \frac{E_x^0}{1-\nu_{xy}^0\nu_{yx}^0}\right]$$

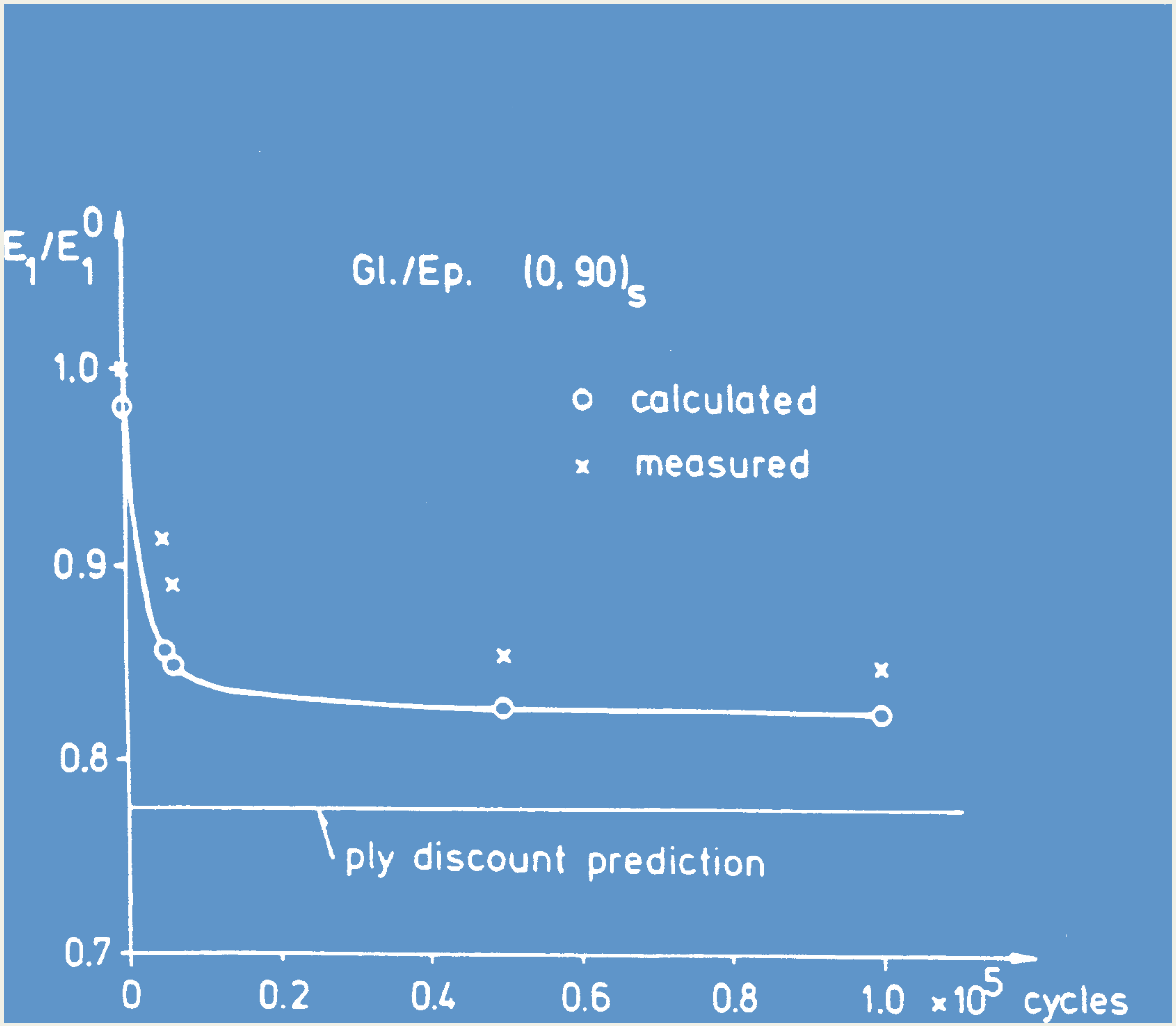
$$a_2 = \frac{s_1 t}{2\kappa t_c^2}\left[\frac{E_y}{1-\nu_{xy}\nu_{yx}} - \frac{E_y^0}{1-\nu_{xy}^0\nu_{yx}^0}\right]$$

$$a_3 = \frac{s_1 t}{2\kappa t_c^2}\left[G_{xy} - G_{xy}^0\right]$$

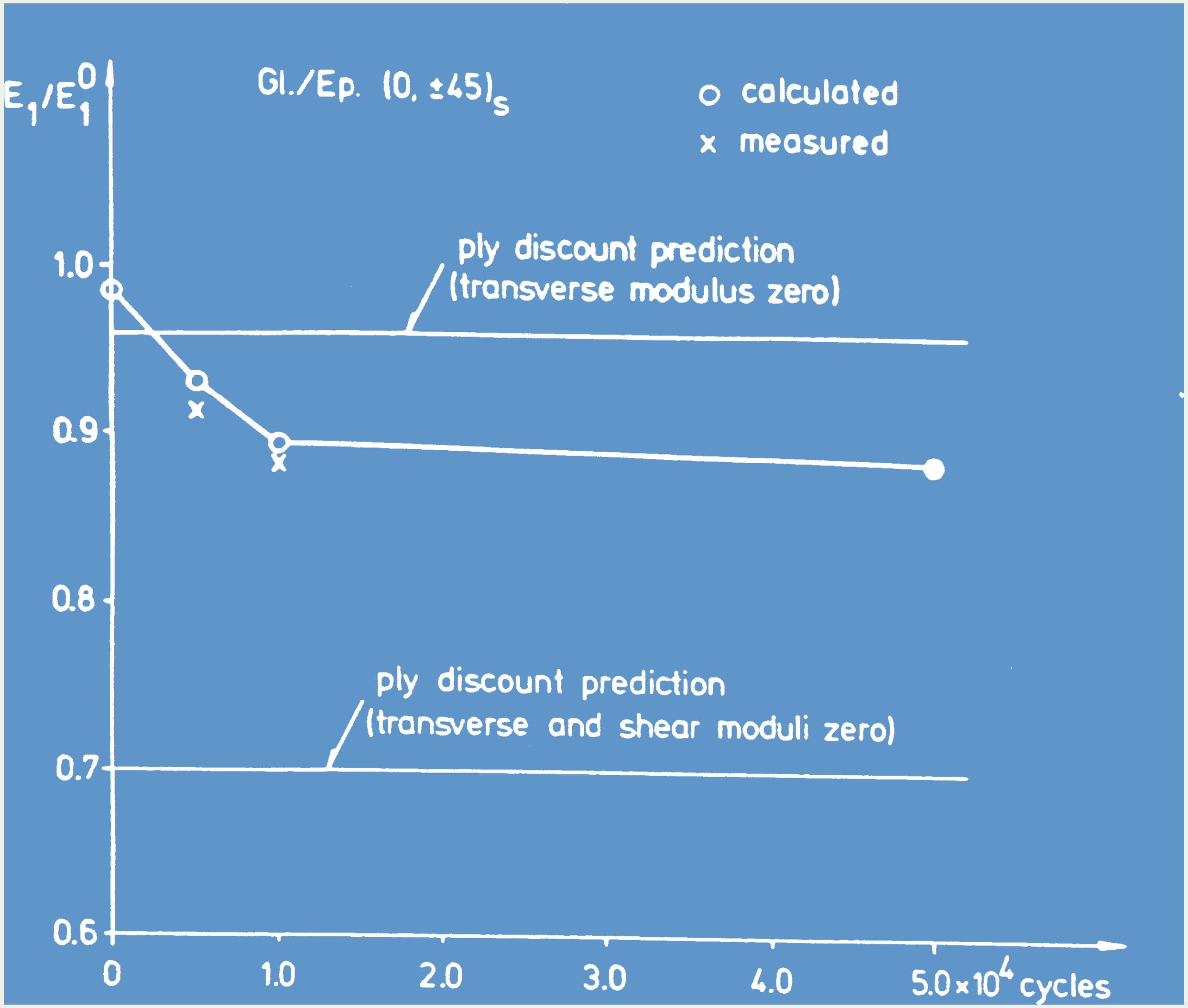
$$a_4 = \frac{s_1 t}{\kappa t_c^2}\left[\frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}} - \frac{\nu_{xy}^0 E_y^0}{1-\nu_{xy}^0\nu_{yx}^0}\right].$$



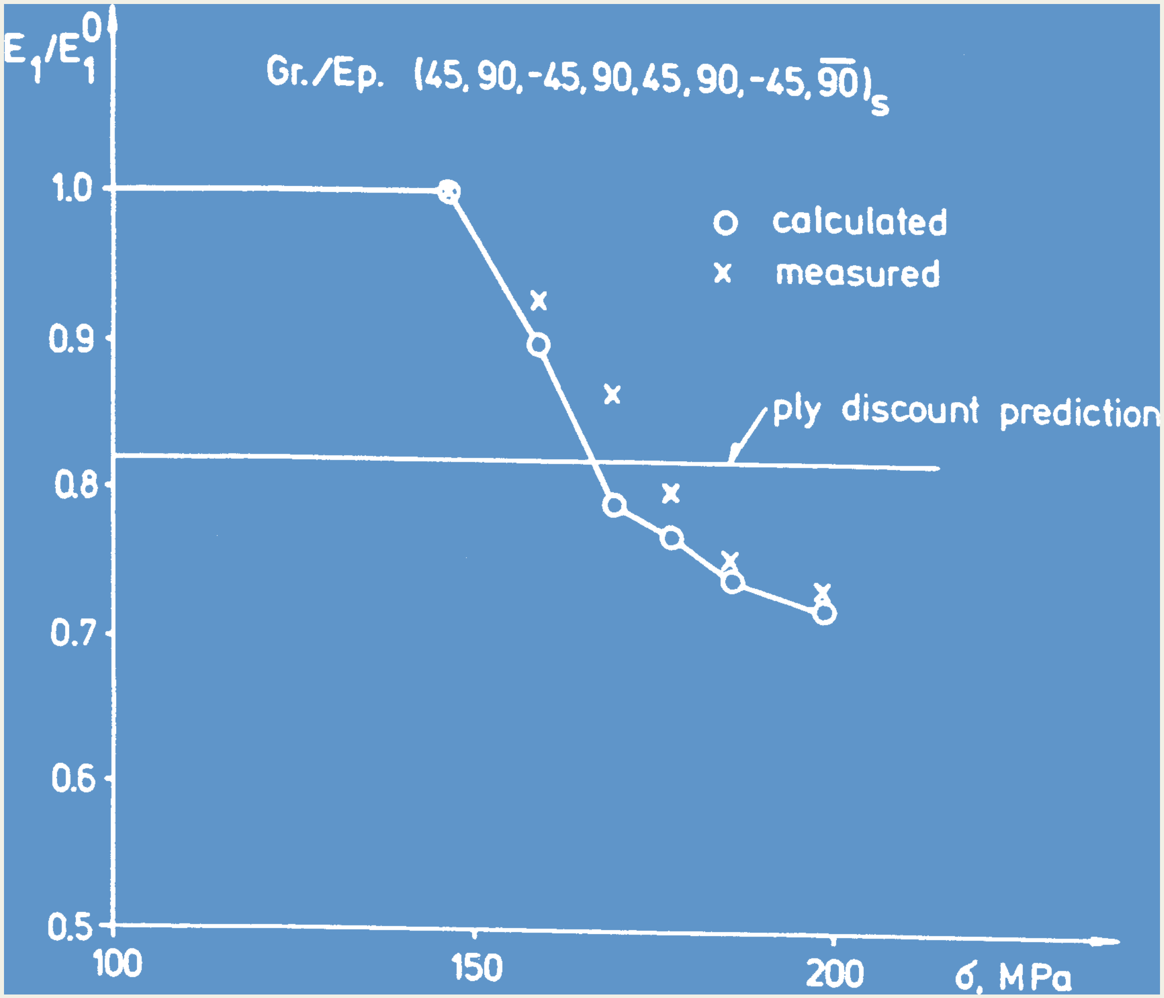
Cross Ply Laminates - More Predictions



Cracks in +45 and -45 Orientations (assumed equal in both orientations)

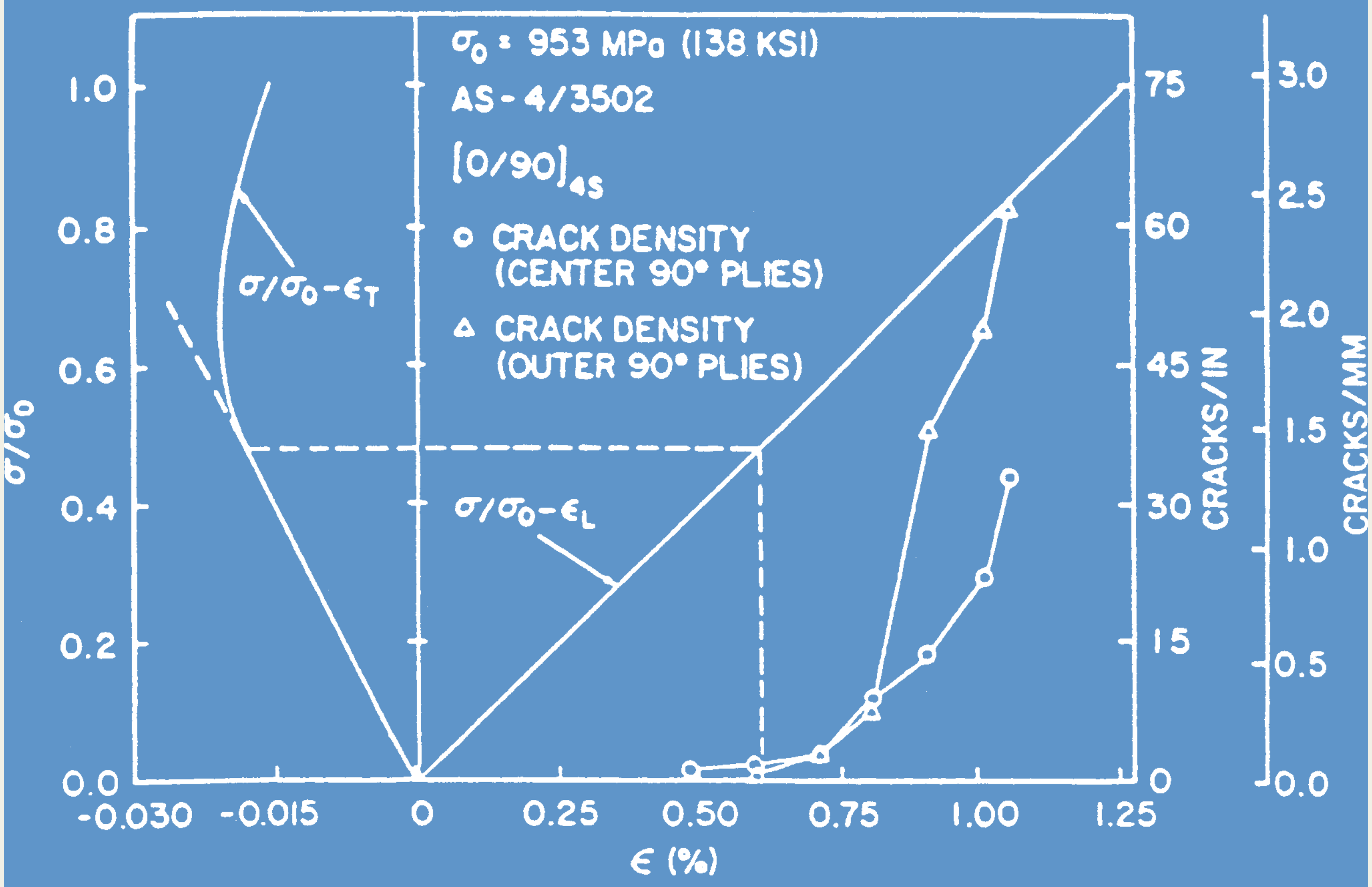


Simultaneous Cracks in 90, +45 and -45 Orientations



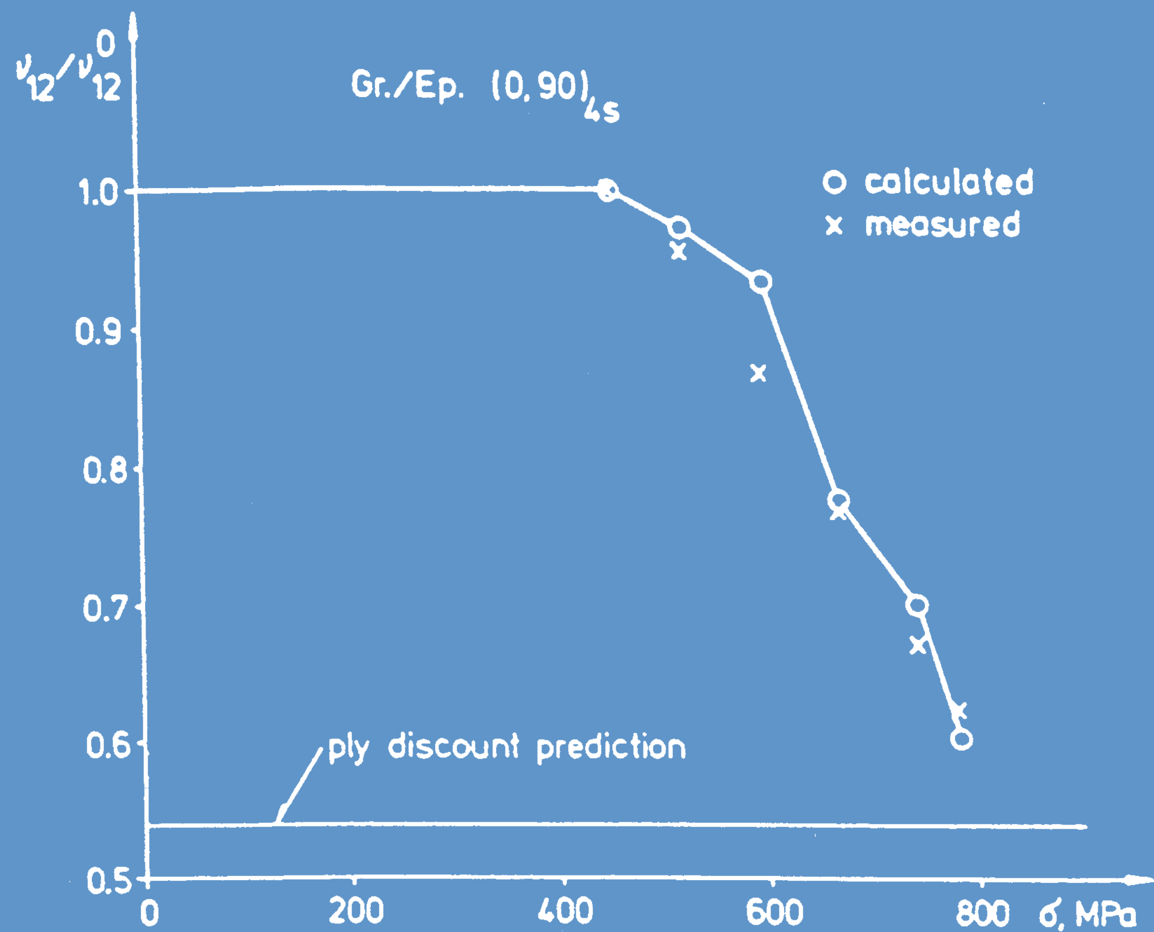
Stiffness Reduction

Monotonic Loading - Cross Ply Laminates

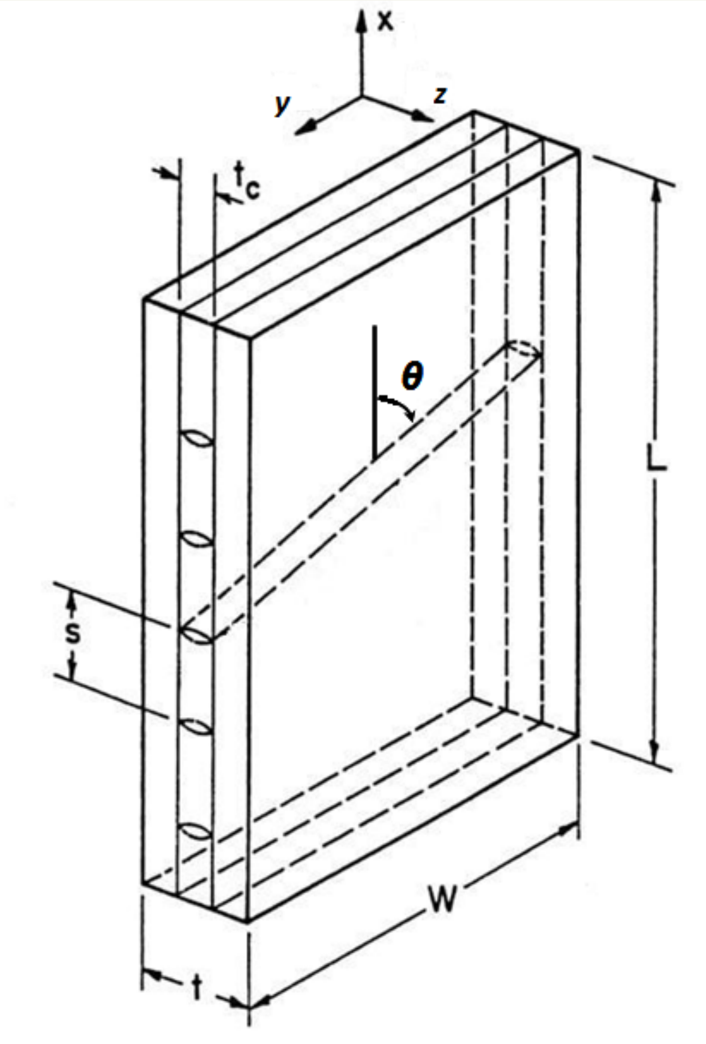


Stiffness Reduction

Monotonic Loading - Cross Ply Laminates



A Problem of Length Scales in Damage



$$D_1 = D_{11}^{(1)} = \frac{\kappa t_c^2}{st} \sin\theta,$$

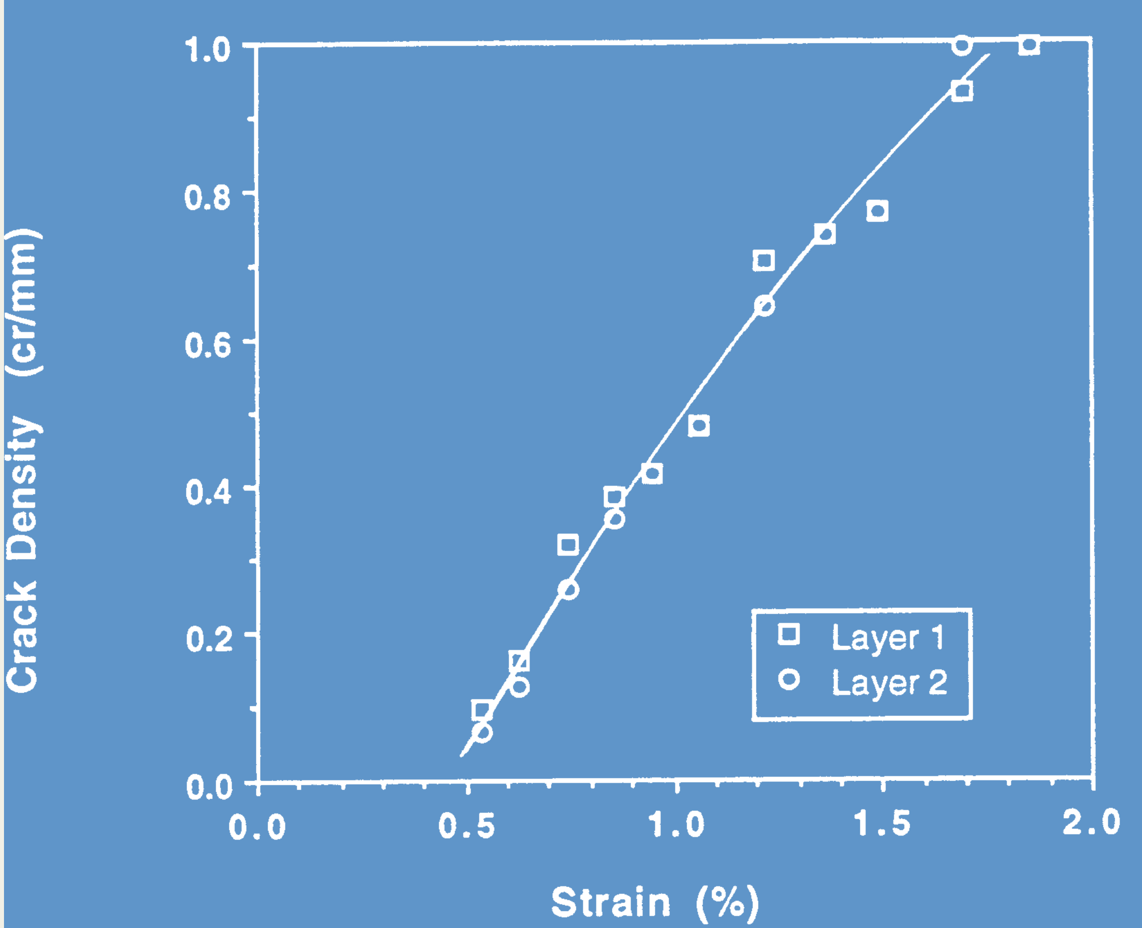
$$D_2 = D_{22}^{(1)} = \frac{\kappa t_c^2}{st} \frac{\cos^2\theta}{\sin\theta},$$

$$D_6 = D_{12}^{(1)} = \frac{\kappa t_c^2}{st} \cos\theta \ .$$

Consider $[0/+\theta_4/-\theta_4/0_{1/2}]_s$ laminates

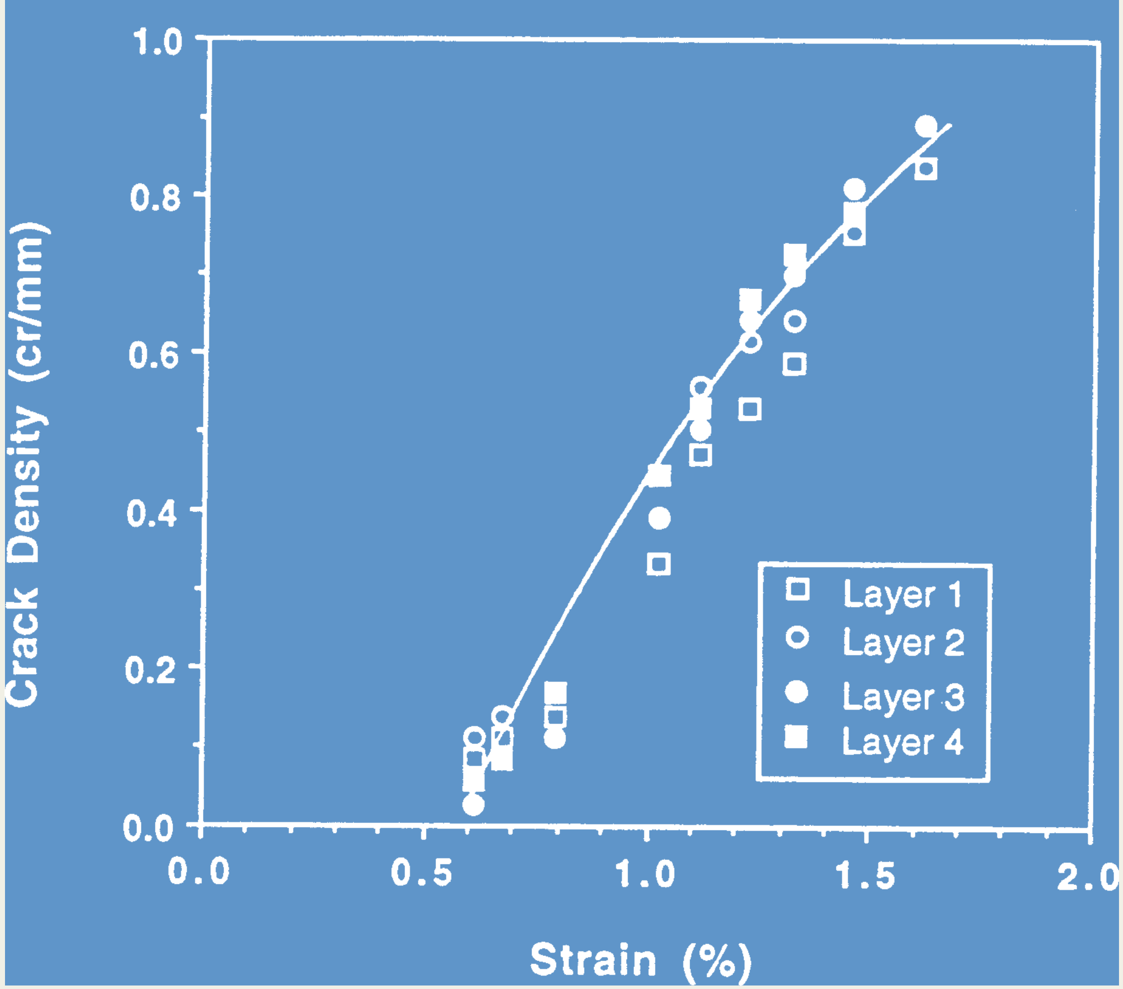
$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

$\theta = 90$



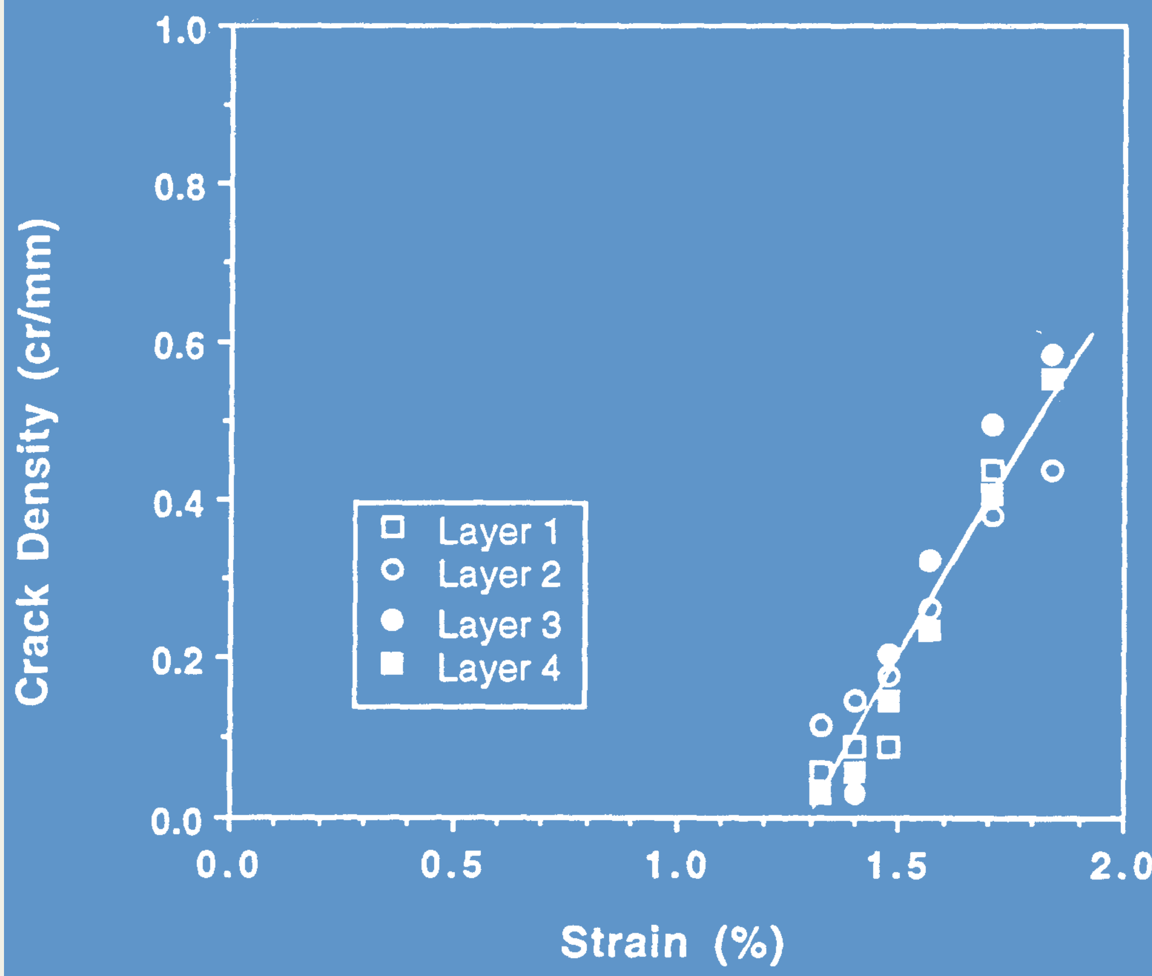
$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

$\theta = 70$



$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

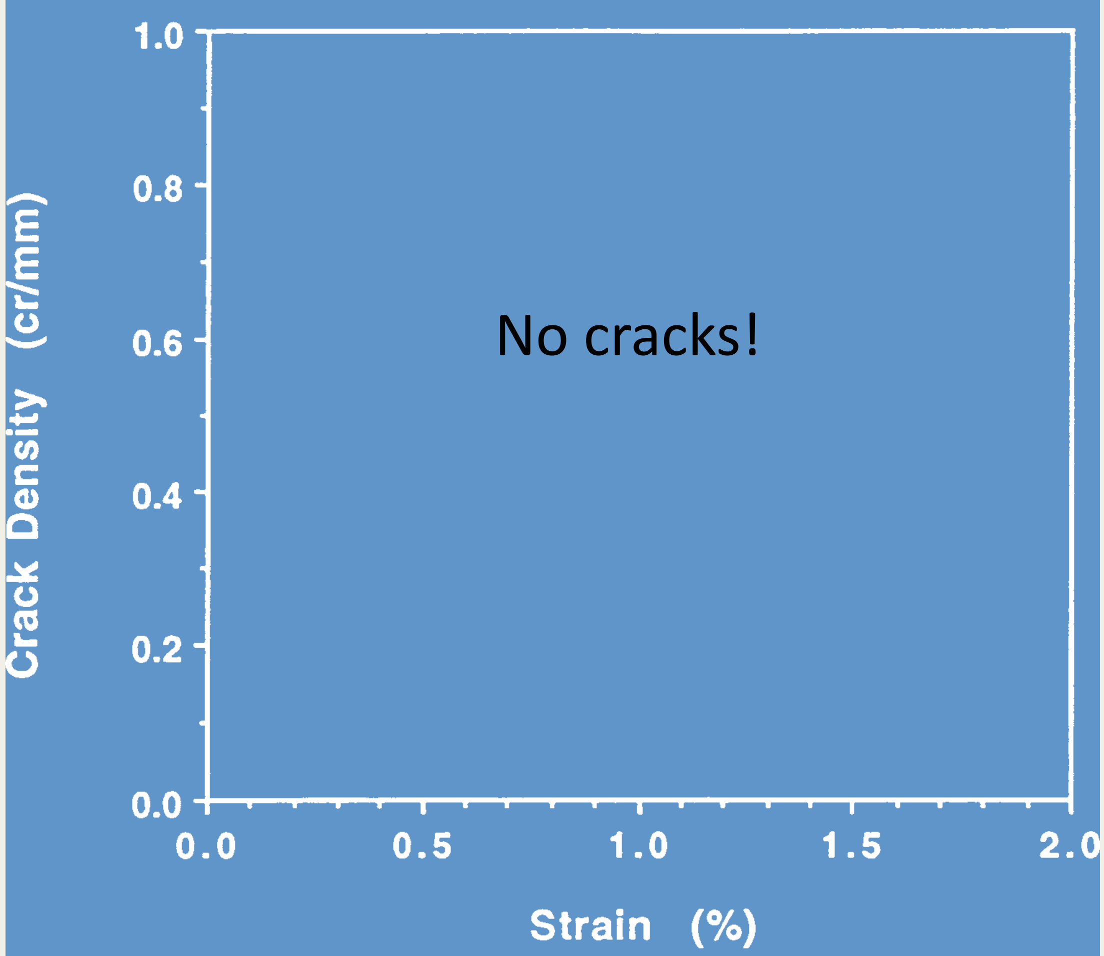
$\theta = 55$



$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

$\theta = 40$

$\theta = 25$

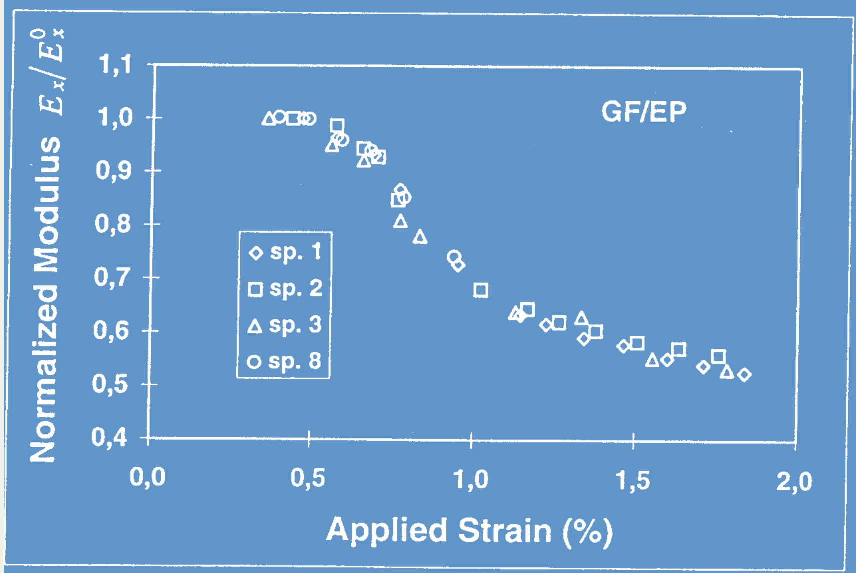


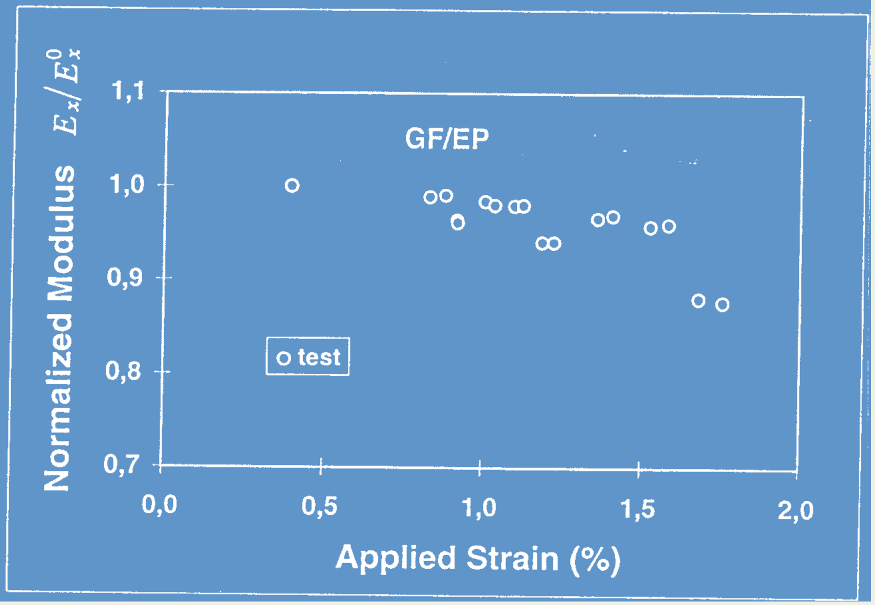
MODULI CHANGES

$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

$\theta = 70$

$\theta = 25$

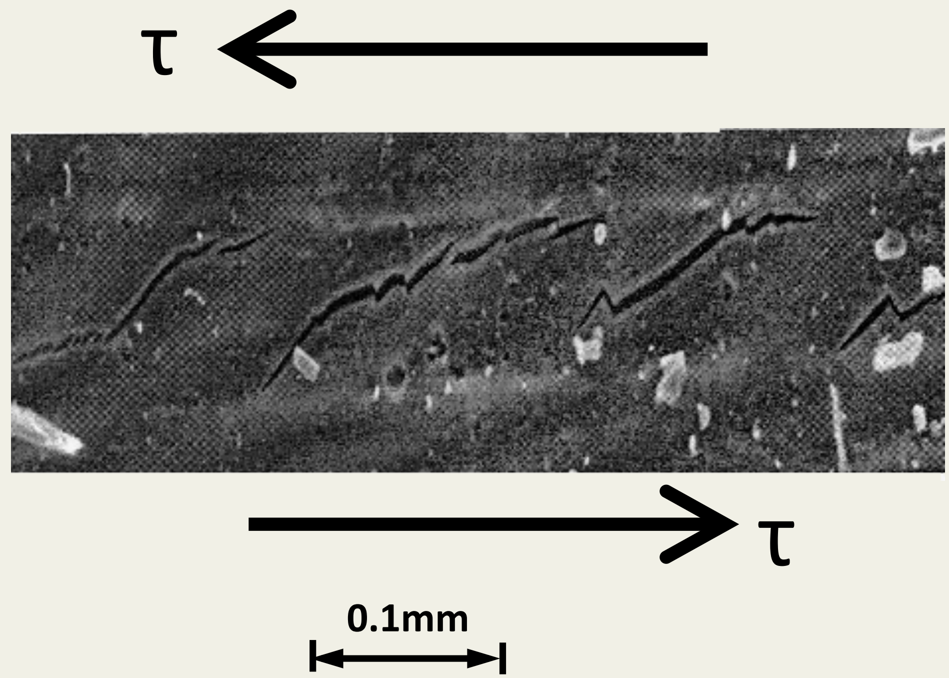




Cracks

No cracks

SHEAR INDUCED CRACKING AT FIBER/MATRIX INTERFACE



Note: The length scale of these cracks is one order of magnitude Smaller than the regular ply cracks

Concluding Remarks on Macro-Damage Mechanics

- The CDM model has the required capabilities of a physically based constitutive framework that can be incorporated into a structural analysis scheme
- The reliance on experimental data for evaluation of material constants is a less attractive feature of the model, particularly for multiple damage modes
- An approach to improve this aspect, known as synergistic damage mechanics, will be discussed next

PART 4:
SYNERGISTIC DAMAGE MECHANICS

Topics

- Multiple Damage Modes
- Synergistic Damage Mechanics
- Damage Evolution

Two Damage Modes

$\psi = \psi_P\left(\varepsilon_{ij}, D_{ij}^{(\alpha)}\right)$ As before for one damage mode, now α = 1 and 2

Irreducible integrity bases (polynomial invariants)
For orthotropic symmetry, two damage modes: (Adkins, 1960)

$$
\begin{aligned}
&\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}^2, \varepsilon_{31}^2, \varepsilon_{12}^2, \varepsilon_{23}\varepsilon_{31}\varepsilon_{12}, \\
&D_{11}^{(1)}, D_{22}^{(1)}, D_{33}^{(1)}, \left(D_{23}^{(1)}\right)^2, \left(D_{31}^{(1)}\right)^2, \left(D_{12}^{(1)}\right)^2, D_{23}^{(1)} D_{31}^{(1)} D_{12}^{(1)}, \\
&D_{11}^{(2)}, D_{22}^{(2)}, D_{33}^{(2)}, \left(D_{23}^{(2)}\right)^2, \left(D_{31}^{(2)}\right)^2, \left(D_{12}^{(2)}\right)^2, D_{23}^{(2)} D_{31}^{(2)} D_{12}^{(2)}, \\
&\varepsilon_{23} D_{23}^{(1)}, \varepsilon_{31} D_{31}^{(1)}, \varepsilon_{12} D_{12}^{(1)}, \varepsilon_{23} D_{23}^{(2)}, \varepsilon_{31} D_{31}^{(2)}, \varepsilon_{12} D_{12}^{(2)}, \\
&\varepsilon_{31}\varepsilon_{12} D_{23}^{(1)}, \varepsilon_{12}\varepsilon_{23} D_{31}^{(1)}, \varepsilon_{23}\varepsilon_{31} D_{12}^{(1)}, \varepsilon_{31}\varepsilon_{12} D_{23}^{(2)}, \varepsilon_{12}\varepsilon_{23} D_{31}^{(2)}, \varepsilon_{23}\varepsilon_{31} D_{12}^{(2)}, \\
&\varepsilon_{23} D_{31}^{(1)} D_{12}^{(1)}, \varepsilon_{31} D_{12}^{(1)} D_{23}^{(1)}, \varepsilon_{12} D_{23}^{(1)} D_{31}^{(1)}, \varepsilon_{23} D_{31}^{(2)} D_{12}^{(2)}, \varepsilon_{31} D_{12}^{(2)} D_{23}^{(2)}, \varepsilon_{12} D_{23}^{(2)} D_{31}^{(2)} .
\end{aligned}
$$

In Voigt notation, for in-plane strains (thin laminates):

$$\varepsilon_1, \varepsilon_2, \varepsilon_6^2$$
$$D_1^{(1)}, D_2^{(1)}, \left(D_6^{(1)}\right)^2, D_1^{(2)}, D_2^{(2)}, \left(D_6^{(2)}\right)^2$$
$$\varepsilon_6 D_6^{(1)}, \varepsilon_6 D_6^{(2)}\ .$$

$$\begin{aligned}
\rho\psi = P_0 &+ \left\{c_1\varepsilon_1^2 + c_2\varepsilon_2^2 + c_3\varepsilon_6^2 + c_4\varepsilon_1\varepsilon_2\right\} \\
&+ \varepsilon_1^2 \left\{c_5 D_1^{(1)} + c_6 D_2^{(1)} + c_7 D_1^{(2)} + c_8 D_2^{(2)}\right\} \\
&+ \varepsilon_2^2 \left\{c_9 D_1^{(1)} + c_{10} D_2^{(1)} + c_{11} D_1^{(2)} + c_{12} D_2^{(2)}\right\} \\
&+ \varepsilon_6^2 \left\{c_{13} D_1^{(1)} + c_{14} D_2^{(1)} + c_{15} D_1^{(2)} + c_{16} D_2^{(2)}\right\} \\
&+ \varepsilon_1\varepsilon_2 \left\{c_{17} D_1^{(1)} + c_{18} D_2^{(1)} + c_{19} D_1^{(2)} + c_{20} D_2^{(2)}\right\} \\
&+ \varepsilon_1\varepsilon_6 \left\{c_{21} D_6^{(1)} + c_{22} D_6^{(2)}\right\} + \varepsilon_2\varepsilon_6 \left\{c_{23} D_6^{(1)} + c_{24} D_6^{(2)}\right\} \\
&+ P_1\left(\varepsilon_p, D_q^{(1)}\right) + P_2\left(\varepsilon_p, D_q^{(2)}\right) + P_3\left(D_q^{(1)}\right) + P_4\left(D_q^{(2)}\right)
\end{aligned}$$

No free energy in virgin state, $P_0 = 0$

Assuming no residual stresses, $P_1 = P_2 = 0$

Stiffness-Damage Relationships

$$C_{pq} = \rho \frac{\partial \psi}{\partial \varepsilon_p \partial \varepsilon_q} \quad \Rightarrow \quad \mathbf{C_{pq}} = \mathbf{C^0_{pq}} + \mathbf{C^{(1)}_{pq}} + \mathbf{C^{(2)}_{pq}}$$

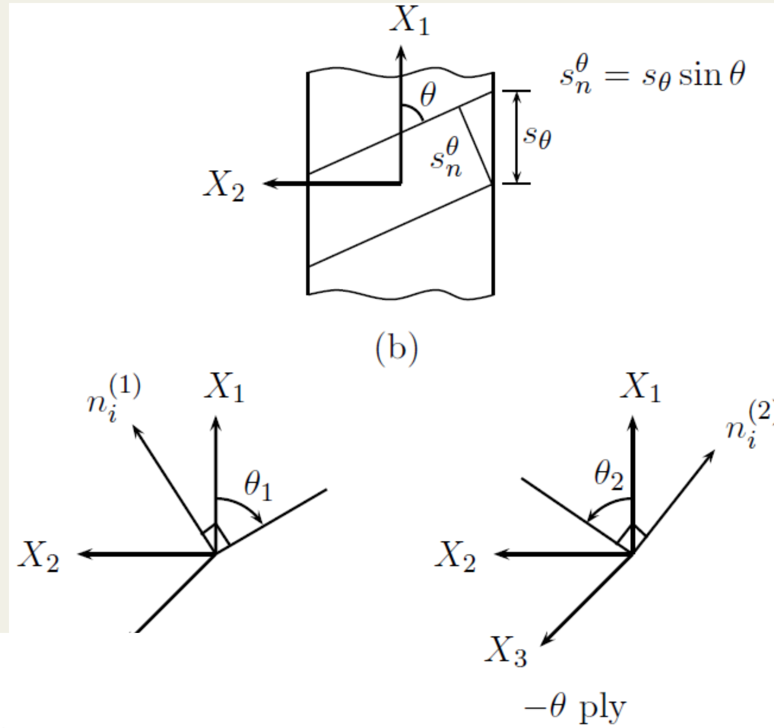
$$\mathbf{C^{(1)}_{pq}} = \begin{bmatrix} 2c_5 D_1^{(1)} + 2c_6 D_2^{(1)} & c_{17} D_1^{(1)} + c_{18} D_2^{(1)} & c_{21} D_6^{(1)} \\ & 2c_9 D_1^{(1)} + 2c_{10} D_2^{(1)} & c_{23} D_6^{(1)} \\ Symm & & 2c_{13} D_1^{(1)} + 2c_{14} D_2^{(1)} \end{bmatrix}$$

$$\mathbf{C^{(2)}_{pq}} = \begin{bmatrix} 2c_7 D_1^{(2)} + 2c_8 D_2^{(2)} & c_{19} D_1^{(2)} + c_{20} D_2^{(2)} & c_{22} D_6^{(2)} \\ & 2c_{11} D_1^{(2)} + 2c_{12} D_2^{(2)} & c_{24} D_6^{(2)} \\ Symm & & 2c_{15} D_1^{(2)} + 2c_{16} D_2^{(2)} \end{bmatrix}.$$

For N damage modes, $$\mathbf{C_{pq}} = \mathbf{C^0_{pq}} + \sum_{\alpha=1}^{N} \mathbf{C^{(\alpha)}_{pq}} .$$

Two Symmetrical Damage Modes:

$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates



$$D_{ij}^{(\alpha)} = \frac{\kappa t_c^2}{s_n^\theta t} n_i n_j \ .$$

$$\alpha = 1 : n_i^{(1)} = (\sin\theta, \cos\theta, 0)$$

$$D_1^{(1)} = \frac{\kappa^{\theta^+} t_c^2}{s_n^{\theta^+} t} \sin^2\theta; \qquad D_2^{(1)} = \frac{\kappa^{\theta^+} t_c^2}{s_n^{\theta^+} t} \cos^2\theta; \qquad D_6^{(1)} = \frac{\kappa^{\theta^+} t_c^2}{s_n^{\theta^+} t} \sin\theta\cos\theta$$

$$\alpha = 2 : n_i^{(2)} = (\sin\theta, -\cos\theta, 0)$$

$$D_1^{(2)} = \frac{\kappa^{\theta^-} t_c^2}{s_n^{\theta^-} t} \sin^2\theta; \qquad D_2^{(2)} = \frac{\kappa^{\theta^-} t_c^2}{s_n^{\theta^-} t} \cos^2\theta; \qquad D_6^{(2)} = -\frac{\kappa^{\theta^-} t_c^2}{s_n^{\theta^-} t} \sin\theta\cos\theta$$

$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

For equal crack spacing in both orientations, $\kappa^{\theta^+} = \kappa^{\theta^-} = \kappa^{\theta};\ \ s_n^{\theta^+} = s_n^{\theta^-} = s_n^{\theta}$.

$$C_{11}^{(1)} + C_{11}^{(2)} = 2\frac{\kappa_\theta t_c^2}{s_n^\theta t}\left[(c_5 + c_7)\sin^2\theta + (c_6 + c_8)\cos^2\theta\right]$$

$$C_{22}^{(1)} + C_{22}^{(2)} = 2\frac{\kappa_\theta t_c^2}{s_n^\theta t}\left[(c_9 + c_{11})\sin^2\theta + (c_{10} + c_{12})\cos^2\theta\right]$$

$$C_{66}^{(1)} + C_{66}^{(2)} = 2\frac{\kappa_\theta t_c^2}{s_n^\theta t}\left[(c_{13} + c_{15})\sin^2\theta + (c_{14} + c_{16})\cos^2\theta\right]$$

$$C_{12}^{(1)} + C_{12}^{(2)} = \frac{\kappa_\theta t_c^2}{s_n^\theta t}\left[(c_{17} + c_{19})\sin^2\theta + (c_{18} + c_{20})\cos^2\theta\right]$$

$$C_{16}^{(1)} + C_{16}^{(2)} = \frac{\kappa_\theta t_c^2}{s_n^\theta t}\sin\theta\cos\theta\left[-c_{21} + c_{22}\right] = 0$$

$$C_{26}^{(1)} + C_{26}^{(2)} = \frac{\kappa_\theta t_c^2}{s_n^\theta t}\sin\theta\cos\theta\left[-c_{23} + c_{24}\right] = 0.$$

$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

$$\mathbf{C}_{\mathbf{pq}}^{(1)} + \mathbf{C}_{\mathbf{pq}}^{(2)} = \begin{bmatrix} 2a_1 D_1 + 2b_1 D_2 & a_4 D_1 + b_4 D_2 & 0 \\ & 2a_2 D_1 + 2b_2 D_2 & 0 \\ Symm & & 2a_3 D_1 + 2b_3 D_2 \end{bmatrix}$$

where

$$a_1 = c_5 + c_7, \quad a_2 = c_9 + c_{11}, \quad a_3 = c_{13} + c_{15}, \quad a_4 = c_{17} + c_{19};$$
$$b_1 = c_6 + c_8, \quad b_2 = c_{10} + c_{12}, \quad b_3 = c_{14} + c_{16}, \quad b_4 = c_{18} + c_{20}$$

(Note: c_i constants are from polynomial expansion of ψ)

Denote:

$$a_1(\theta) = a_1 \sin^2\theta + b_1 \cos^2\theta,$$
$$a_2(\theta) = a_2 \sin^2\theta + b_2 \cos^2\theta,$$
$$a_3(\theta) = a_3 \sin^2\theta + b_3 \cos^2\theta,$$
$$a_4(\theta) = a_4 \sin^2\theta + b_4 \cos^2\theta.$$

$$\mathbf{C}_{\mathbf{pq}}^{(1)} + \mathbf{C}_{\mathbf{pq}}^{(2)} = D_\theta \begin{bmatrix} 2a_1(\theta) & a_4(\theta) & 0 \\ & 2a_2(\theta) & 0 \\ Symm & & 2a_3(\theta) \end{bmatrix}$$

$$\boxed{D_\theta = \frac{\kappa_\theta t_c^2}{s_n^\theta t}.}$$

Note: The constraint parameter is dependent on θ

Assessment of CDM for Multiple Damage Modes

- Experimental evaluation of the constraint parameter for each mode is not practical
- Computational determination of crack surface separation is needed, from which the constraint parameter can be evaluated
- This input from computational (or analytical, if possible) micromechanics in CDM produces a powerful approach called Synergistic Damage Mechanics (SDM)

Multiscale Synergistic Damage Mechanics

COMPUTATIONAL MICROMECHANICS

Determine COD and constraint parameter(s)

$$\beta = \frac{\kappa_\theta}{\kappa_{90}} = \frac{\left(\overline{\Delta u_2}\right)_{\pm\theta_n}}{\left(\overline{\Delta u_2}\right)_{90_8}} \; ; \; \left(\overline{\Delta u_2}\right)_{\pm\theta_n} = \left(\overline{\Delta u_2}\right)_{+\theta_n} + \left(\overline{\Delta u_2}\right)_{-\theta_n}$$

Structural scale: Micro

EXPERIMENTAL/ COMPUTATIONAL

Evaluate damage constants using available data for reference laminate configuration $[0/90_8/0_{1/2}]_s$

SYNERGISTIC DAMAGE MECHANICS

Use SDM to determine stiffness reduction in present laminate configuration $[0_m/\pm\theta_n/0_{m/2}]_s$

Structural scale: Meso

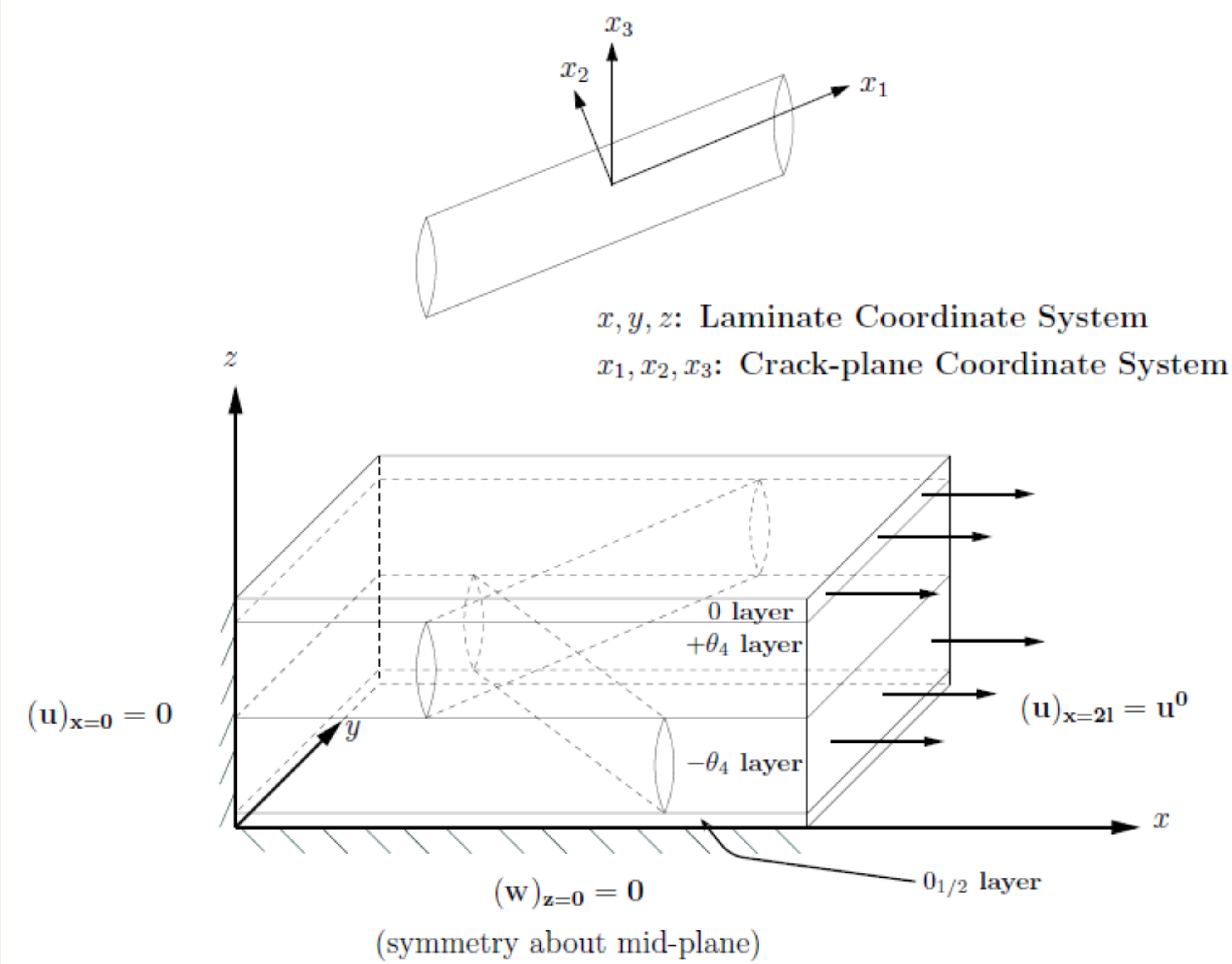
STRUCTURAL ANALYSIS

Analyze overall structural response to external loading using the reduced stiffness properties

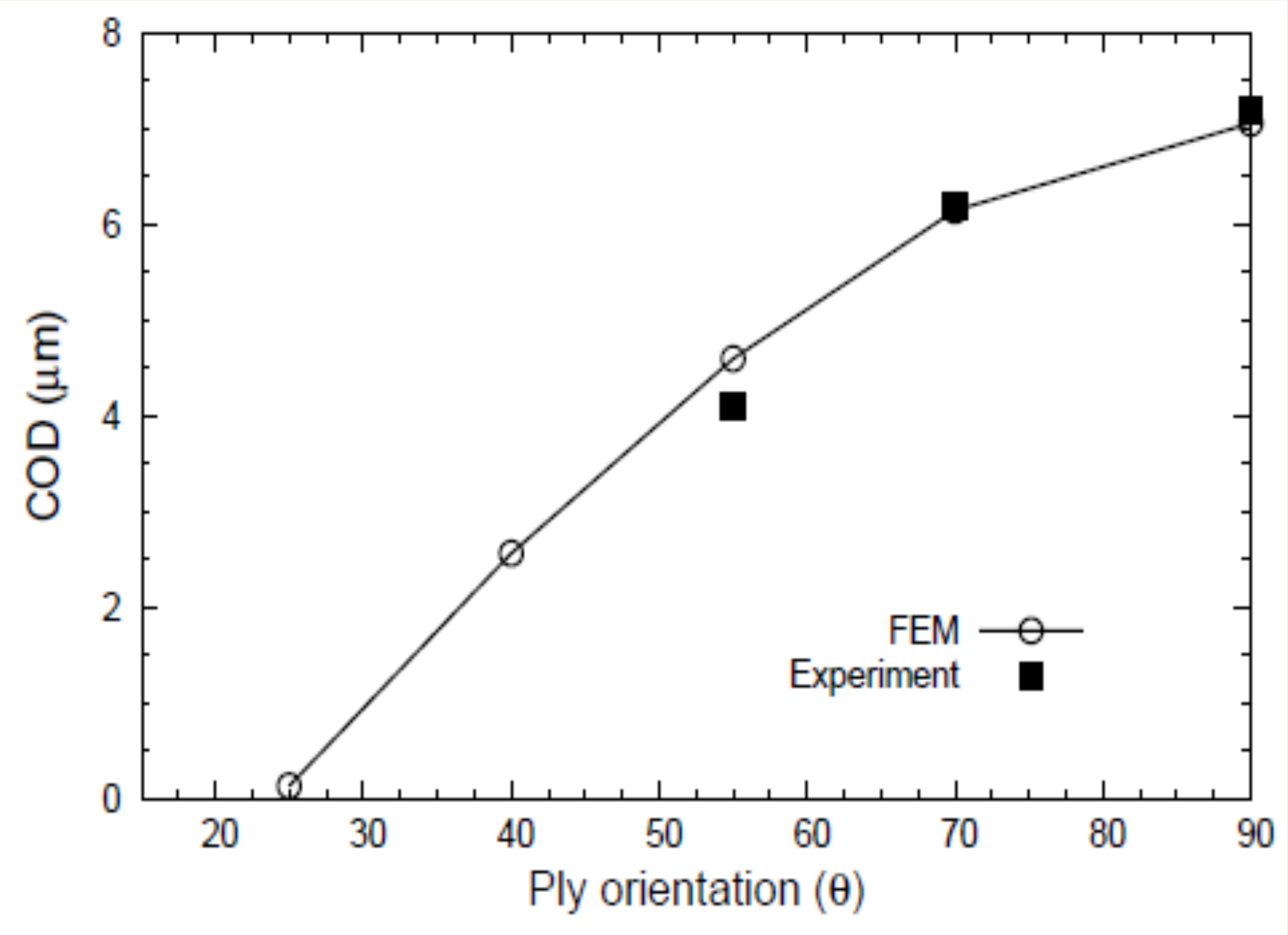
Structural scale: Macro

Note the use of a reference laminate

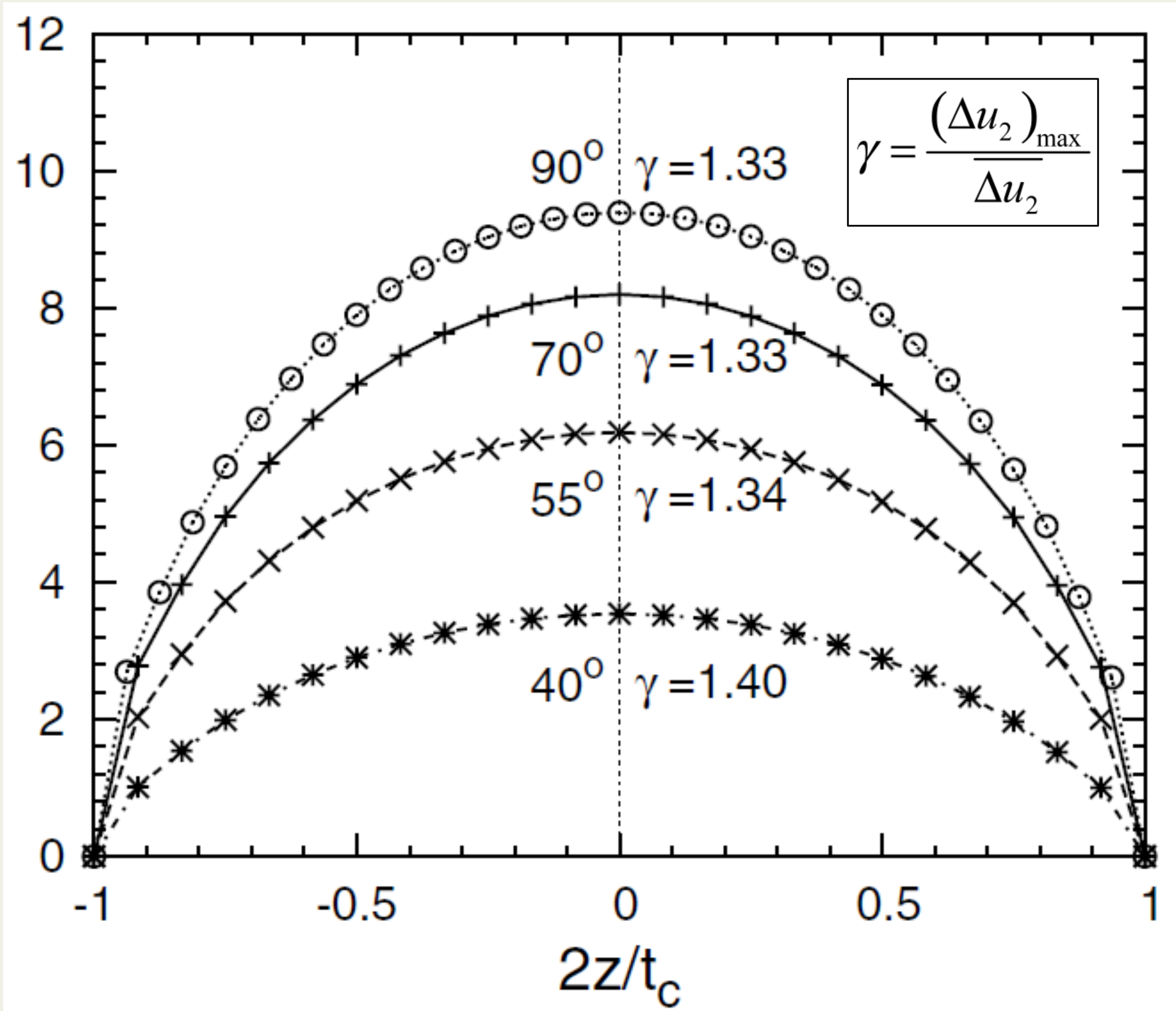
Finite Elements Evaluation of Crack Opening Displacement (COD) for $[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates



Calculated COD Compared With Experiments



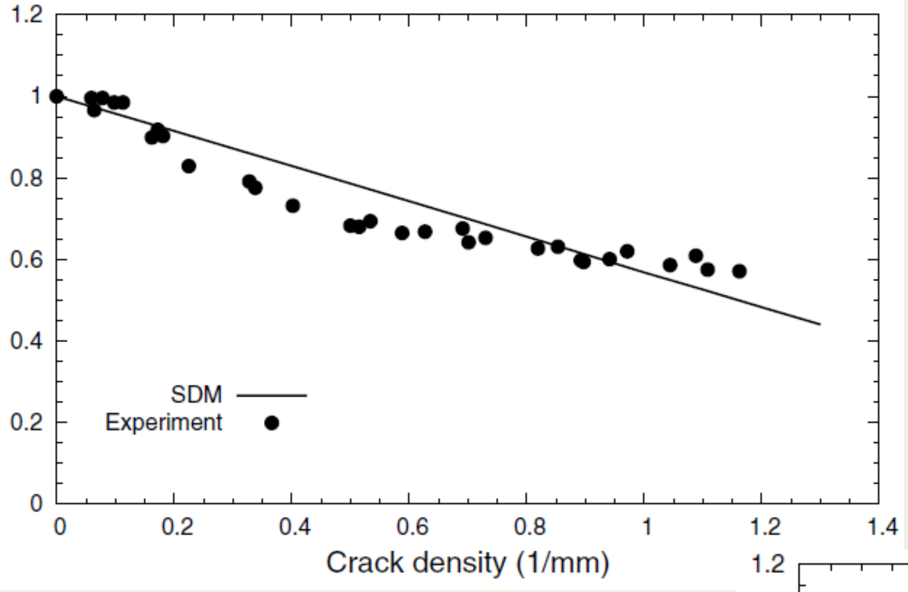
Change in COD With Constraint



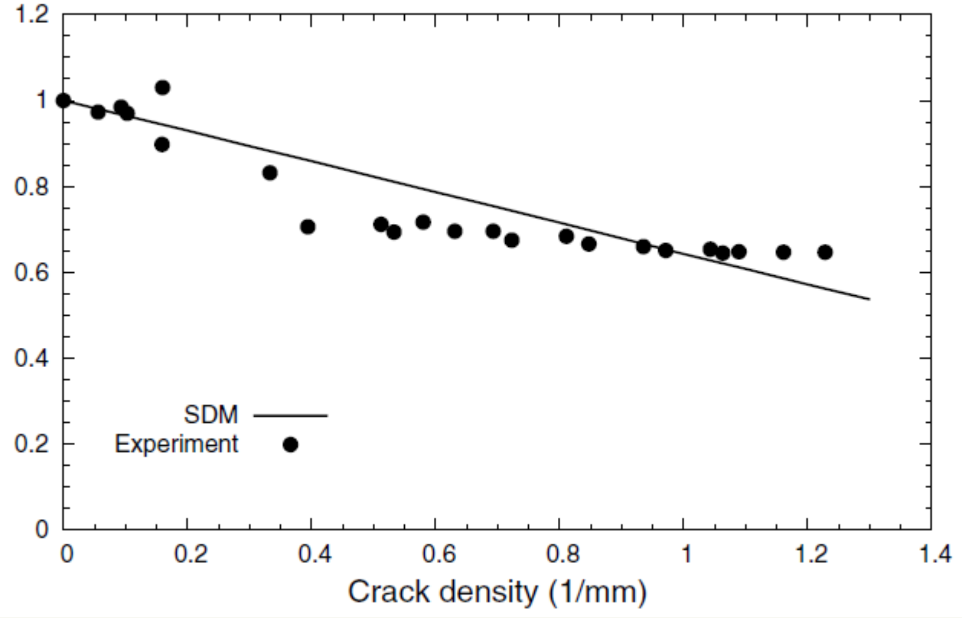
SDM Predictions for $[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

θ=70°

$$\frac{E_x}{E_x^0}$$



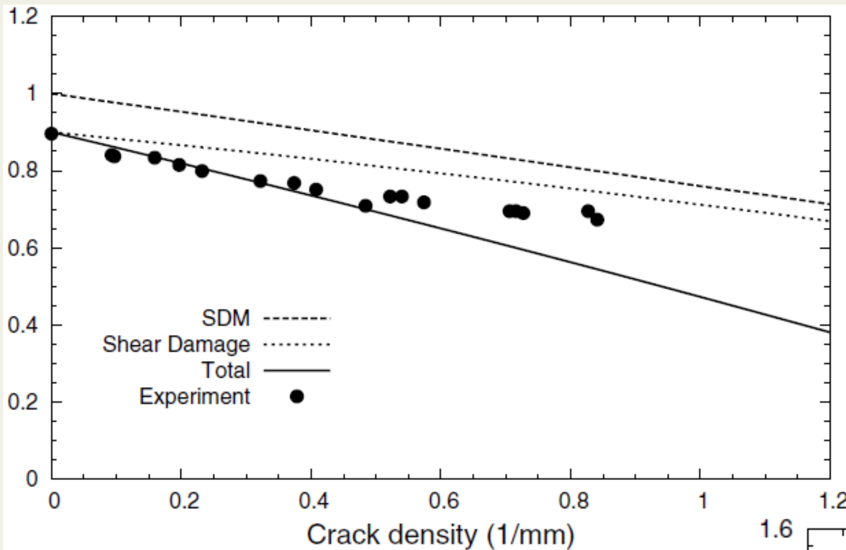
$$\frac{\nu_{xy}}{\nu_{xy}^0}$$



SDM Predictions for$[0/+\theta_4/-\theta_4/0_{1/2}]_s$ Laminates

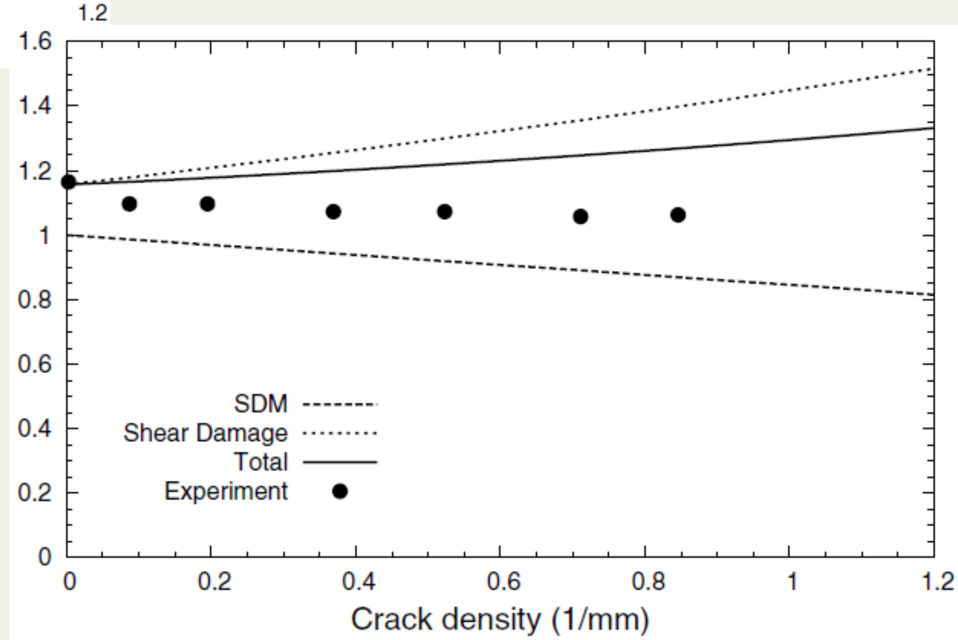
θ=55°

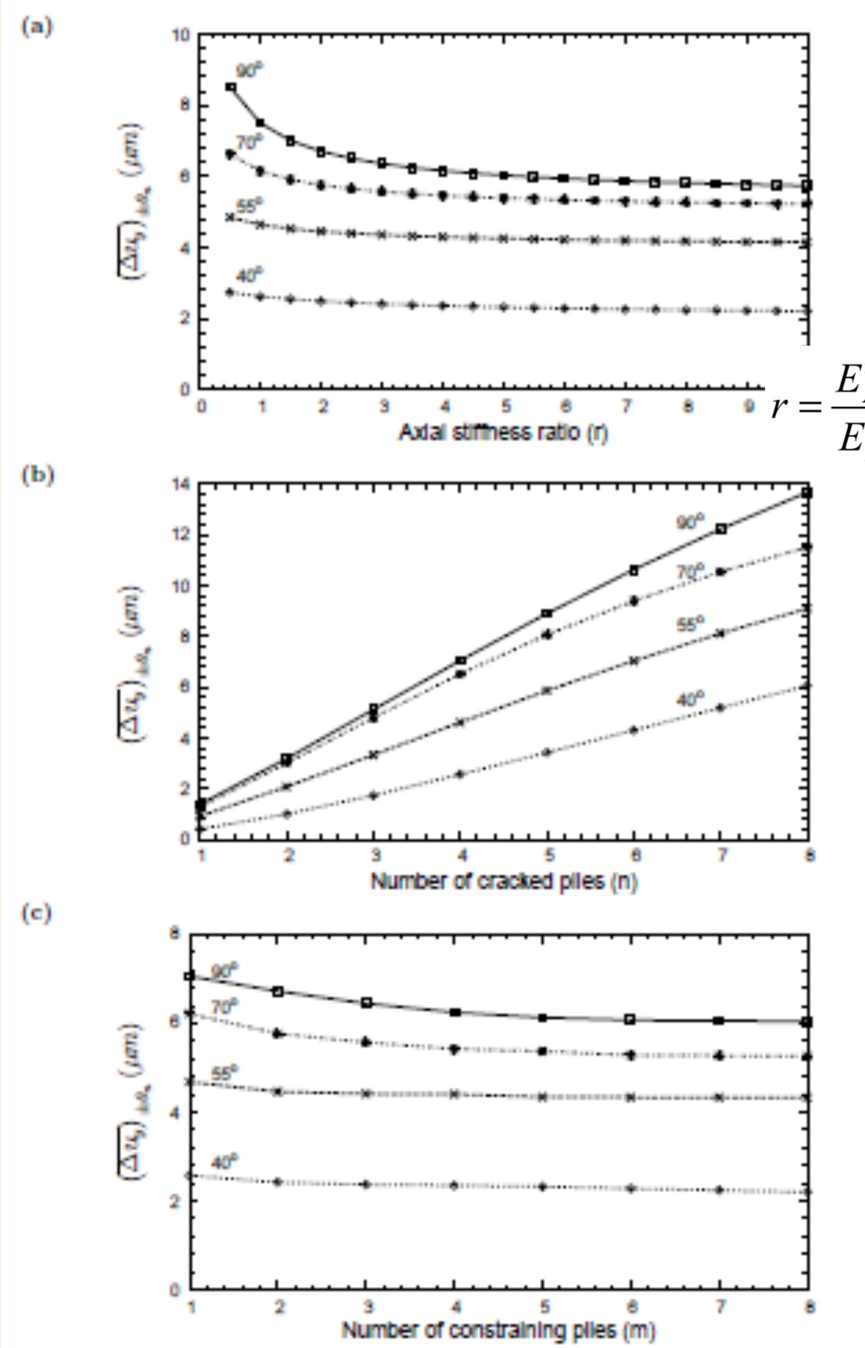
$$\frac{E_x}{E_x^0}$$



Note: Shear damage (small scale cracks) caused by significant shear stress in the θ-plies

$$\frac{\nu_{xy}}{\nu_{xy}^0}$$





$$r = \frac{E_A^{\pm\theta}}{E_A^{90}}$$

Parametric Study of COD In $[0_m/\pm\theta_n/0_{m/2}]_s$ Laminates

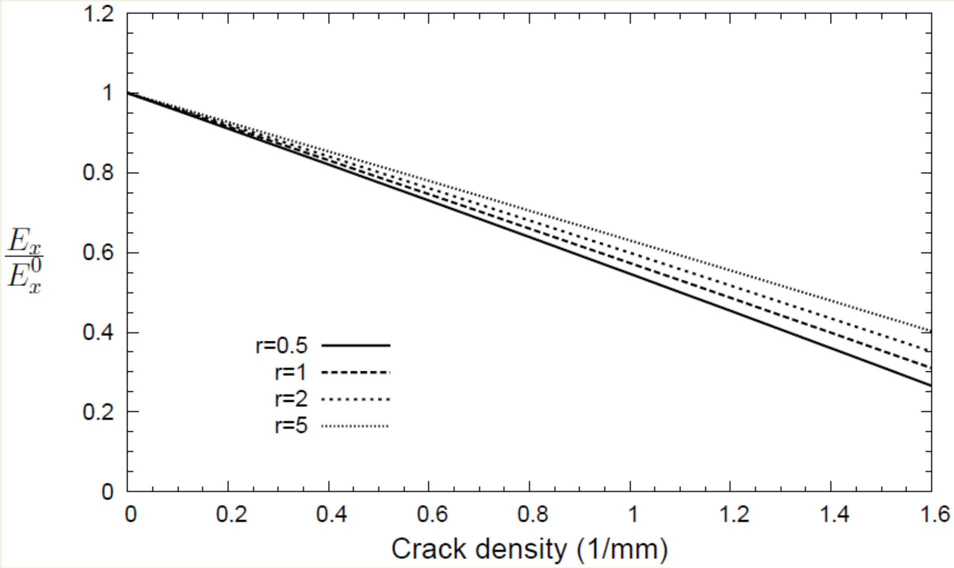
$$\left(\overline{\Delta u_2}\right)_{\pm\theta_n} = U.f_1(\theta).f_2(r).f_3(m).f_4(n)$$

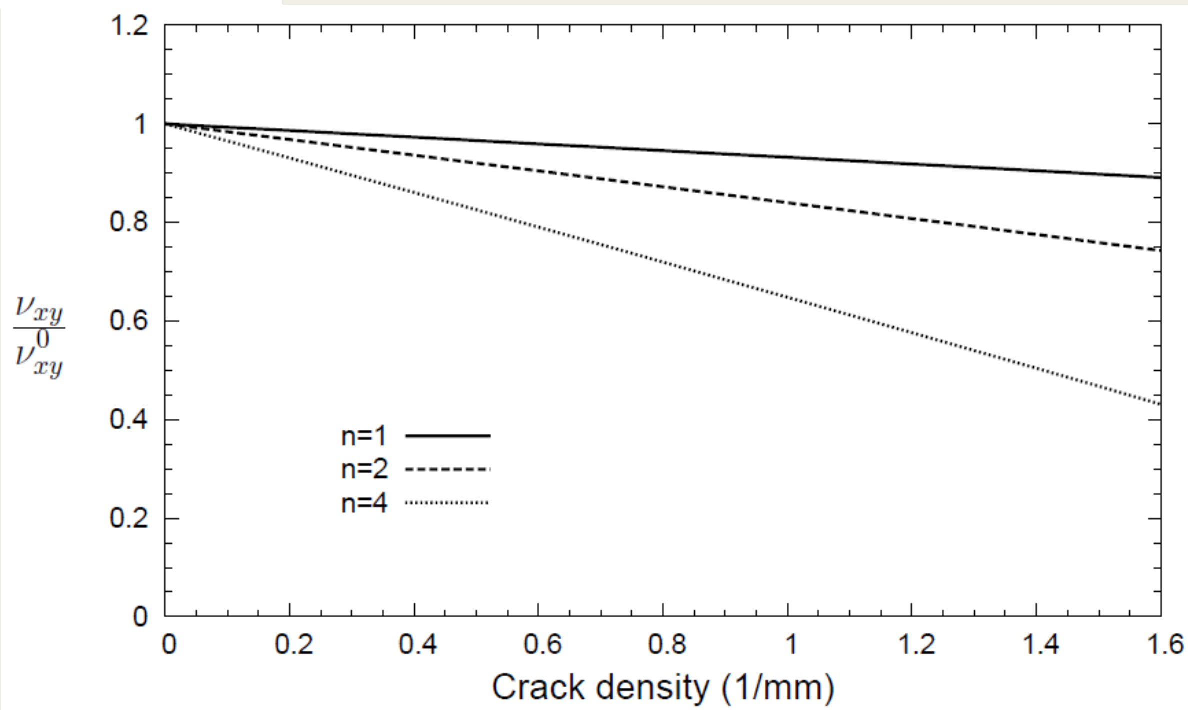
U : COD of $[0/90_8/0_{1/2}]_s$ laminate

$$f_1(\theta) = \sin^2\theta$$
$$f_2(r) = r^{-c_1}$$
$$f_3(m) = \frac{c_2}{m} + c_3$$
$$f_4(n) = c_4 n^{c_5} \ .$$

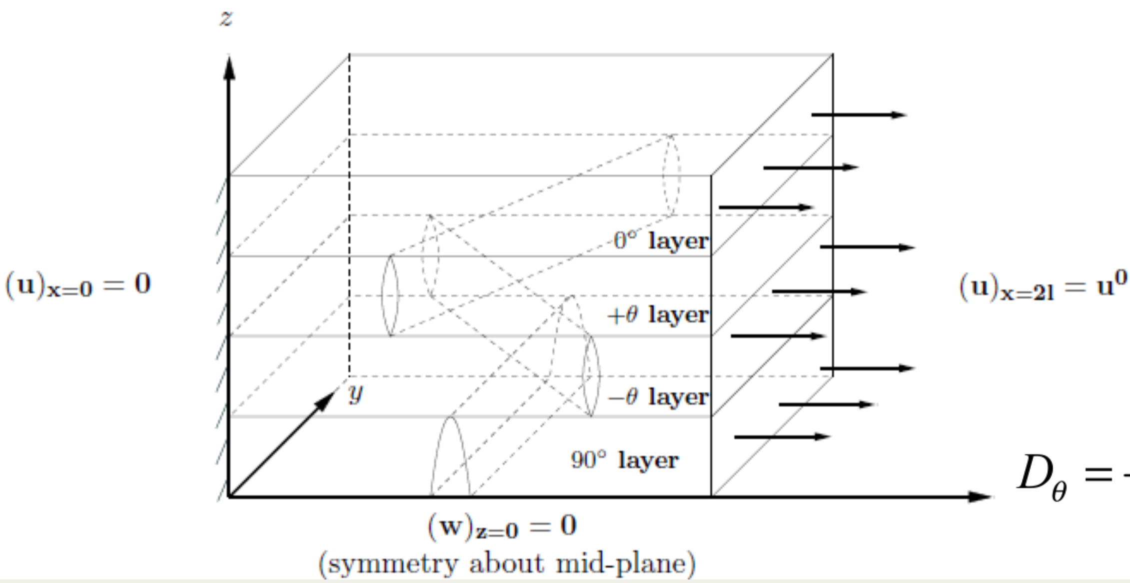
c_i: curve-fitting constants

SDM Predictions of Properties Using the COD Equation





Three Damage Modes: Cracking in θ, -θ, and 90° Plies



$$D_1^{(3)} = \frac{\kappa_{90} t_{90}^2}{s^{90} t}, \quad D_2^{(3)} = D_6^{(3)} = 0 \,.$$

$$\mathbf{C}_{\mathbf{pq}} = \mathbf{C}_{\mathbf{pq}}^{\mathbf{0}} + \mathbf{C}_{\mathbf{pq}}^{(1)} + \mathbf{C}_{\mathbf{pq}}^{(2)} + \mathbf{C}_{\mathbf{pq}}^{(3)}$$

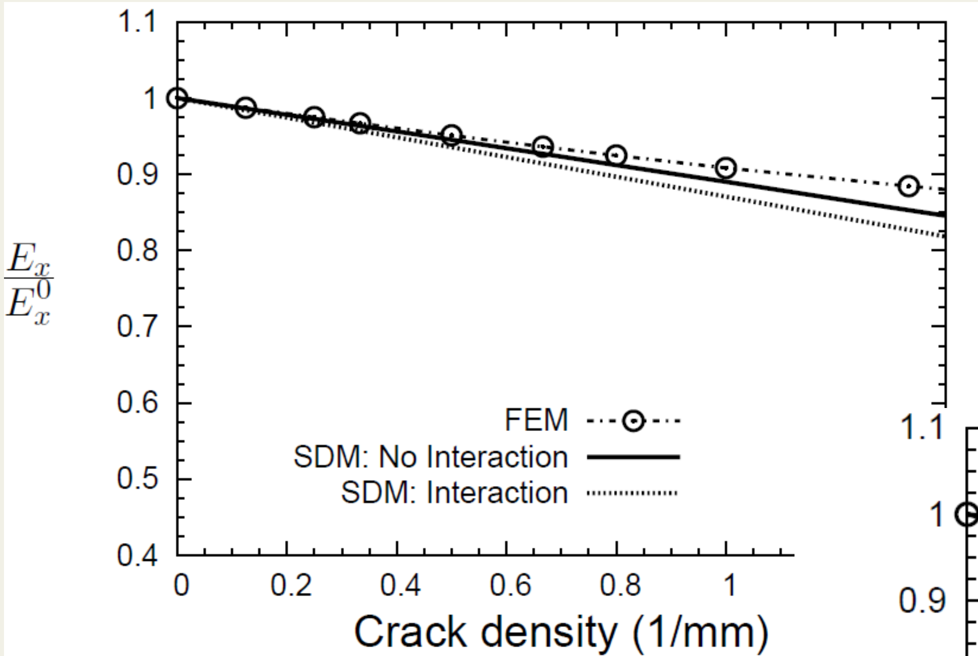
$$D_\theta = \frac{\kappa_\theta \left(2nt_0\right)^2}{s_n^\theta t}; \qquad D_{90} = \frac{\kappa_{90} \left(2rt_0\right)^2}{s^{90} t}$$

$[0_m/\pm\theta_n/90_r]_s$ laminate

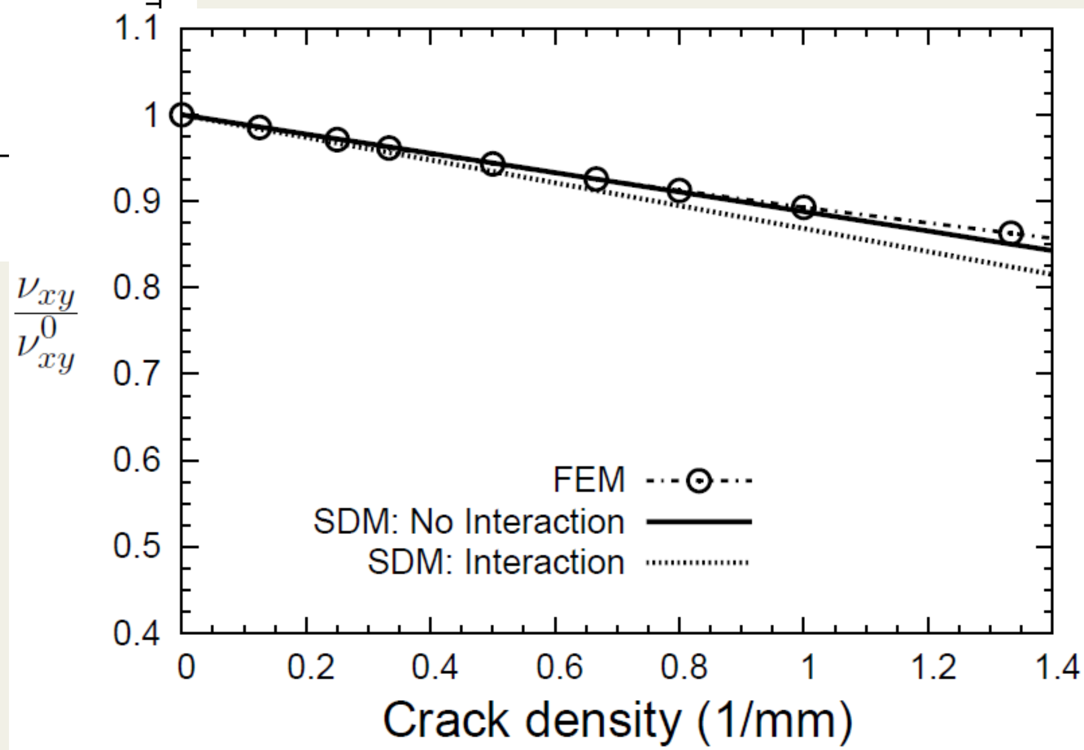
$$\kappa_\theta = \frac{\left(\overline{\Delta u_y}\right)_{\pm\theta_{2n}}}{2nt_0}; \qquad \kappa_{90_{4n+2r}} = \frac{\left(\overline{\Delta u_y}\right)_{90_{4n+2r}}}{(4n+2r)t_0}; \qquad \kappa_{90} = \frac{\left(\overline{\Delta u_y}\right)_{90_{2r}}}{2rt_0} \,.$$

$$\Delta\mathbf{C}_{\mathbf{pq}} = 2D_\theta \begin{bmatrix} 2a_1 & a_4 & 0 \\ & 2a_2 & 0 \\ Symm & & 2\,a_3 \end{bmatrix} + D_{90} \begin{bmatrix} 2a_1' & a_4' & 0 \\ & 2a_2' & 0 \\ Symm & & 2\,a_3' \end{bmatrix}$$

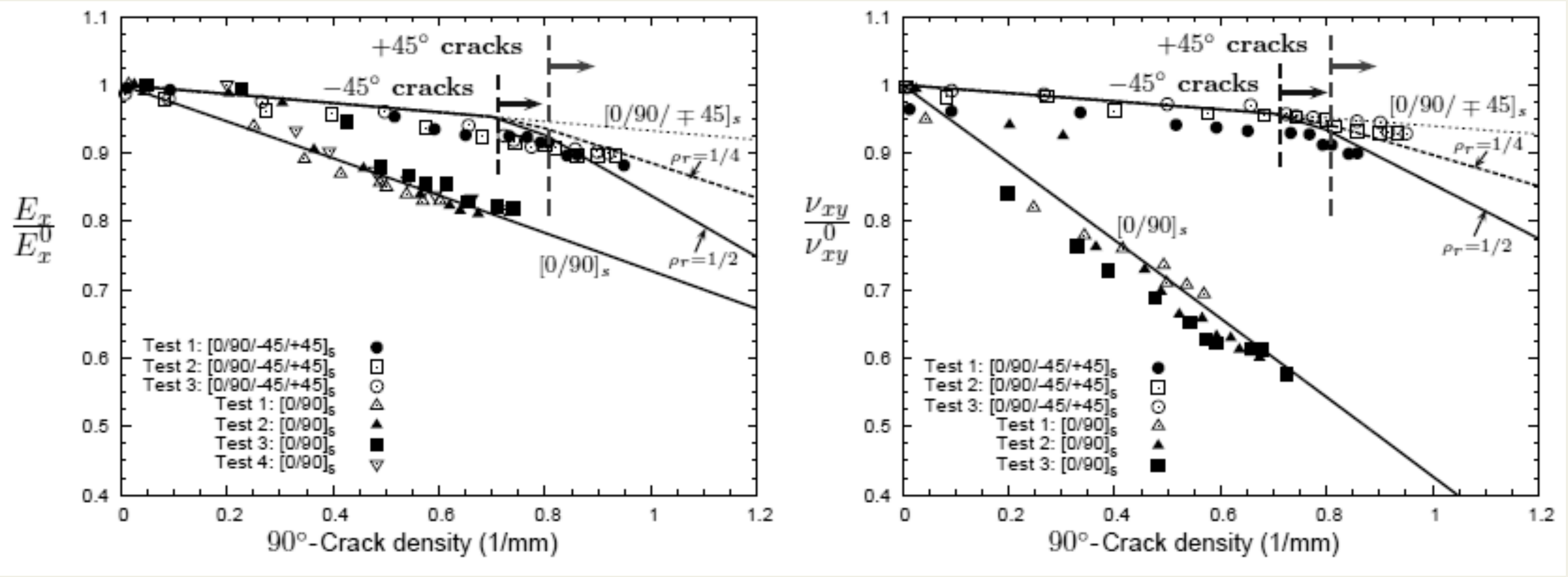
SDM Predictions Compared With FEM



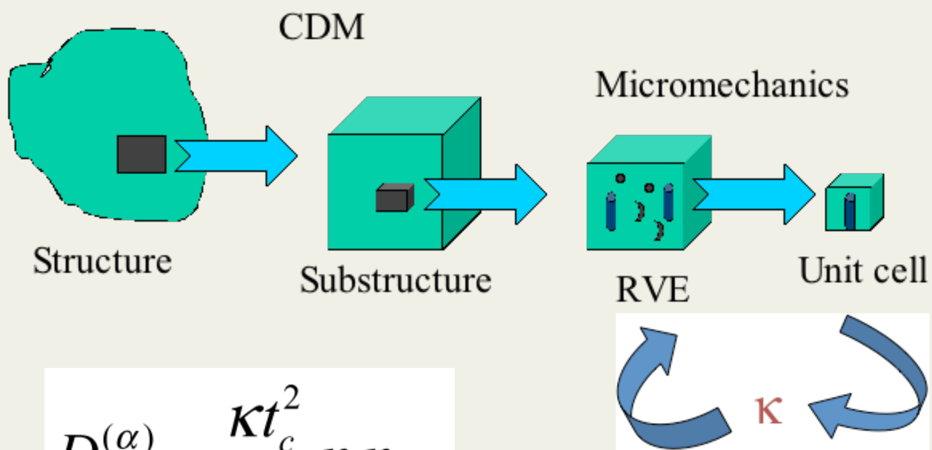
$[0/\pm55/90]_s$ glass/epoxy laminates



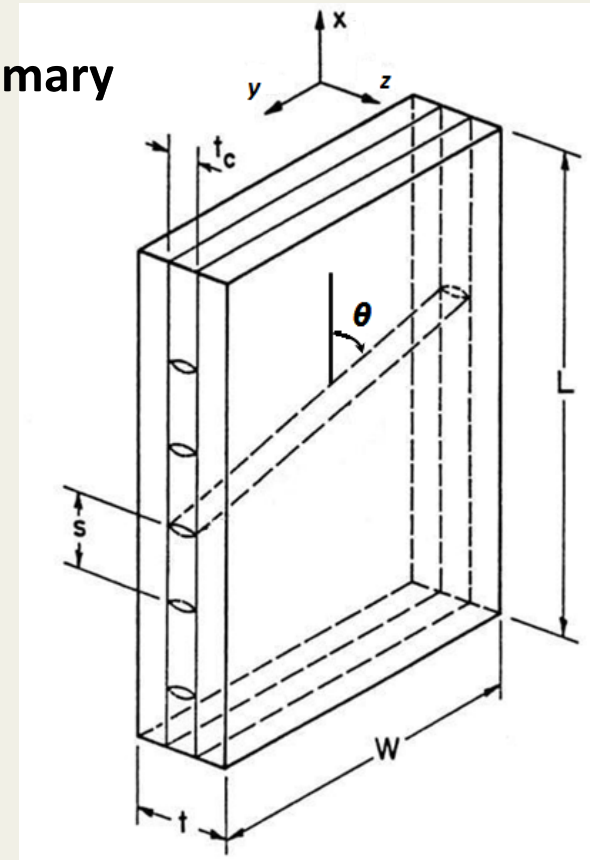
SDM Predictions Compared With Experimental Data



Multiscale Synergistic Damage Mechanics : Summary



$$D_{ij}^{(\alpha)} = \frac{\kappa t_c^2}{s_n^{\theta} t} n_i n_j \ .$$



Residual Response at a damage state: $\frac{R}{R_0} = 1 - \kappa \frac{t_c}{s} f_1 f_2 - (OT)$

R_0 : Initial value
f_1, f_2 : Dimensionless functions of laminate properties and geometry
OT : Other terms that become significant at high crack densities

Future Direction: Incorporation in Structural Analysis